Analysis and numerical experiments on extreme waves through oblique interaction of solitary waves

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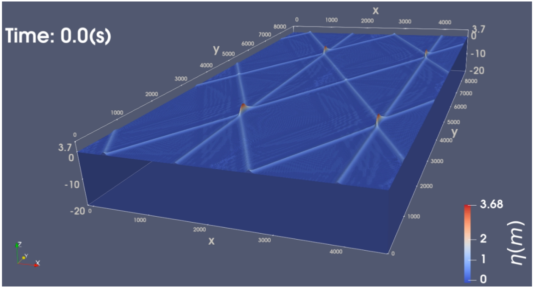
Leeds Institute for Fluid Dynamics; EGU2022: Extreme events in sea waves



Outline

- Motivation
- Mathematical models: Benney-Luke and KP equations
- Exact solutions of the Kadomtsev-Petviashvili equation
  - Examples with one, two, or three line solitons
  - Proof of maximum amplification
- Numerical simulations
  - Two interacting line solitons
  - Three interacting line solitons
- Summary of main results

Motivation on modelling extremely high water waves

- Origin 2010 *bore-soliton-splash*:
- To what extent do exact but idealised extreme- or rogue-wave solutions survive in more realistic settings?
- Will such extreme waves fall apart due to dispersion or other mechanisms?
- Use fourfold and ninefold amplifications of interacting solitons/cnoidal waves.
- What do you think: will we be able to reach the ninefold wave amplification in more realistic calculations or in reality?

Motivation on modelling extremely high water waves

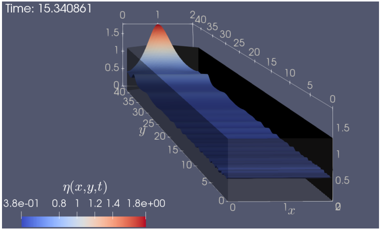
- Rogue waves: anomalously high-amplitude waves

  Abnormality index: $\quad \mathrm{AI} = \dfrac{H_r}{H_s} > 2$

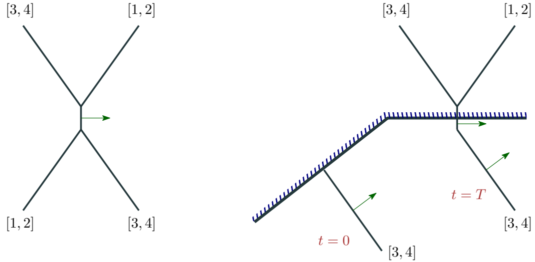
- Large waves at sea occur rarely and are difficult to predict.
- Crossing seas: short-crested sea states stemming from interacting waves with two or three distinct main directions.
- Rogue-wave statistics required but here we focus on deterministic solutions.

Introduction to this work

- Analytical solutions of Kadomtsev-Petviashvili (KP) equation
  - ⇝ line solitons: 2D extension of soliton solutions of KdV
  - ⇝ web solitons: interacting line solitons with different orientations in the far field
  - ⇝ cnoidal waves: periodic short-crested high-amplitude waves
- Resolve these solutions numerically in higher-order wave models, here the Benney-Luke (BL) equations



Introduction to this work

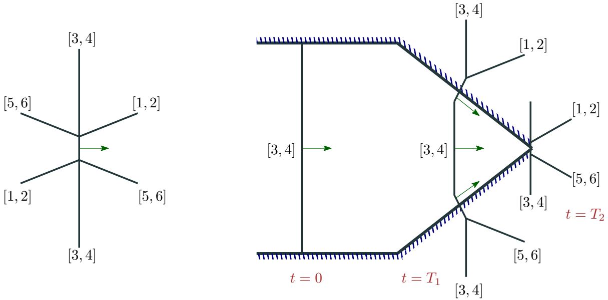


Two-soliton solutions of the KP (amplitude $\tilde{A}$) on infinite plane approximately describe the interaction of one soliton travelling along a wall, then encountering and interacting with a corner

$\rightsquigarrow$ *Maximum amplif.: $4\tilde{A}$* [Miles, 1977; Kodama, 2010; Gidel et al. 2017]

But ... unidirectional KP equation cannot have walls

Introduction to this work

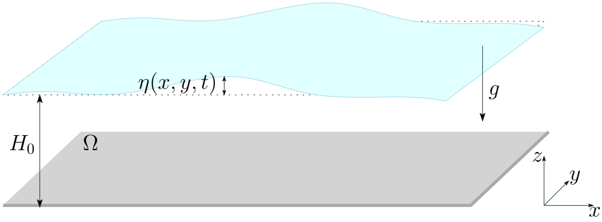


Three-soliton solutions of the KP (amplitude $\tilde{A}$) on infinite plane approximately describe the interaction of one soliton travelling along a channel with two walls and a contraction [Bokhove et al., 2019]
$\rightsquigarrow$ localised soliton splash of max amplification $\sim 8.6\tilde{A}$ [Kodama, 2013]

Here will prove $9\tilde{A}$ max amplification and establish validity in BL

Mathematical hierarchy: potential-flow theory

Velocity potential $\phi(x, y, z, t)$, defined by $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ (irrotational flow)



Water-wave equations

$$\boldsymbol{\nabla}^2\phi = 0 \quad \text{in} \quad \Omega$$

$$\partial_t\eta + \nabla\phi \cdot \nabla\eta - \partial_z\phi = 0 \quad \text{at} \quad z = H_0 + \eta$$

$$\partial_t\phi + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + g\eta = 0 \quad \text{at} \quad z = H_0 + \eta$$

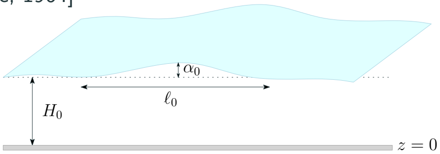
$$\mathbf{n} \cdot \nabla\phi = 0 \quad \text{on} \quad z = 0 \ \text{ and } \ \partial\Omega$$

Mathematical hierarchy: BL and KP approximations

⇝ Shallow water approximation: long wave length compared to mean water depth

⇝ Boussinesq approximation: includes weak dispersive effects

- KdV equation: wave propagation in 1D [Korteweg & de Vries, 1895]
- KP equation: unidirectional propagation in 2DH [Kadomtsev & Petviashvili, 1970]
- Benney-Luke equations: bidirectional propagation in 2DH [Benney & Luke, 1964]



$\epsilon = \alpha_0/H_0 \ll 1$

$\mu = (H_0/\ell_0)^2 \ll 1$

Expansion about the sea-bed potential $\Phi(x, y, t) = \phi(x, y, z = 0, t)$, in powers of the small parameter μ [Pego & Quintero, 1999]

Benney-Luke (BL) equations

$$\partial_t \eta - \frac{\mu}{2}\partial_t \nabla^2 \eta + \nabla\cdot\big((1+\epsilon\eta)\,\nabla\Phi\big) - \frac{2\mu}{3}\nabla^4\Phi = 0 \quad \text{in } \Omega$$

$$\partial_t \Phi - \frac{\mu}{2}\partial_t \nabla^2 \Phi + \frac{\epsilon}{2}\,|\nabla\Phi|^2 + \eta = 0 \quad \text{in } \Omega$$

$$\mathbf{n}\cdot\nabla\Phi = 0 \quad \text{on } \partial\Omega$$

$$\mathbf{n}\cdot\nabla(\nabla^2\Phi) = 0 \quad \text{on } \partial\Omega$$

Total Energy

$$E(t) = \int_\Omega \left(\frac{1}{2}\eta^2 + \frac{1}{2}(1+\epsilon\eta)|\nabla\Phi|^2 + \frac{\mu}{3}(\nabla^2\Phi)^2\right)\,\mathrm{d}x\,\mathrm{d}y$$

is conserved in time due to the Hamiltonian nature of the system

[Bokhove & Kalogirou, 2016]

Kadomtsev-Petviashvili (KP) equation

The KP equation can be obtained from the Benney-Luke equations by introducing the formal perturbation expansions

$$\eta = \tilde{u} + \mathcal{O}(\epsilon^2), \qquad \mathit{\Phi} = \sqrt{\epsilon}\left(\tilde{\Psi} + \mathcal{O}(\epsilon^2)\right),$$

using the transformations

$$X = \sqrt{\frac{\epsilon}{\mu}}\left(\frac{3}{\sqrt{2}}\right)^{1/3}(x-t), \qquad Y = \sqrt{\epsilon}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{3}{\sqrt{2}}\right)^{2/3} y,$$

$$\tau = \epsilon\sqrt{\frac{2\epsilon}{\mu}}\,t, \qquad u = \left(\frac{3}{4}\right)^{1/3}\tilde{u},$$

and taking $\mu = \epsilon^2$, resulting in the KP equation in "standard" form

$$\boxed{\partial_X\left(4\partial_\tau u + 6u\partial_X u + \partial_{XXX}u\right) + 3\partial_{YY}u = 0}$$

This equation includes weak effects in the y-direction.

Exact solution of the KP equation

Web and line-soliton solutions can be constructed using Hirota's transformation

$$u(X,Y,\tau) = 2\partial_{XX} \ln K(X,Y,\tau) = \frac{2\partial_{XX}K}{K} - 2\left(\frac{\partial_X K}{K}\right)^2,$$

where function $K(X,Y,\tau)$ can be obtained from the Wronskian

$$K(X,Y,\tau) = \begin{vmatrix} f_1 & f_1^{(1)} & \dots & f_1^{(N-1)} \\ f_2 & f_2^{(1)} & \dots & f_2^{(N-1)} \\ \vdots & \vdots & & \vdots \\ f_N & f_N^{(1)} & \dots & f_N^{(N-1)} \end{vmatrix}.$$

Particular soliton solutions are obtained by taking [Kodama, 2010]

$$f_i = \sum_{j=1}^{M} a_{ij}\, \mathrm{e}^{\theta_j}, \quad \text{where } \theta_j = k_j X + k_j^2 Y - k_j^3 \tau,$$

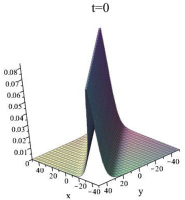
with coefficients k_j being ordered as $k_1 < k_2 < \dots < k_M$. This solution is called a (N_-, N_+)-soliton, comprising line solitons in the far-field $Y \to \pm\infty$.

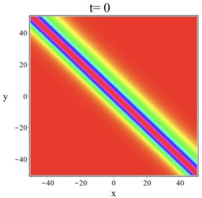
Example: single line soliton

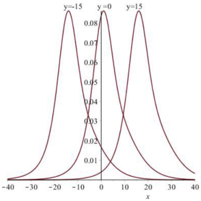
Single line solitons have $(N, M) = (1, 2)$, resulting in $K = f_1 = e^{\theta_1} + e^{\theta_2}$ and the line soliton solution is

$$
\begin{aligned}
u(X, Y, \tau) &= \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1 - \theta_2) \\
&= \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2 \frac{1}{2}\big((k_1 - k_2)X + (k_1^2 - k_2^2)Y - (k_1^3 - k_2^3)\tau\big).
\end{aligned}
$$

The soliton amplitude is $\tilde{A} = \frac{1}{2}(k_1 - k_2)^2$ and its centreline is found by setting the sech^2 argument to zero.







Example: two interacting line solitons

Two line solitons have $(N, M) = (2, 4)$, also called $(2, 2)$-solitons or O-solitons, obtained with functions $f_1 = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}$, $f_2 = \mathrm{e}^{\theta_3} + \mathrm{e}^{\theta_4}$, and

$$K(X, Y, \tau) = (k_3 - k_1)\mathrm{e}^{\theta_1+\theta_3} + (k_3 - k_2)\mathrm{e}^{\theta_2+\theta_3} + (k_4 - k_1)\mathrm{e}^{\theta_1+\theta_4} + (k_4 - k_2)\mathrm{e}^{\theta_2+\theta_4}.$$

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In the far field $Y \to \pm\infty$, we find the single line solitons

$$u_{[1,2]}(X, Y, \tau) = \frac{1}{2}(k_2 - k_1)^2 \mathrm{sech}^2 \frac{1}{2}(\theta_1 - \theta_2 - \ln a),$$
$$u_{[3,4]}(X, Y, \tau) = \frac{1}{2}(k_4 - k_3)^2 \mathrm{sech}^2 \frac{1}{2}(\theta_3 - \theta_4 - \ln b),$$

where a, b depend on k_j.

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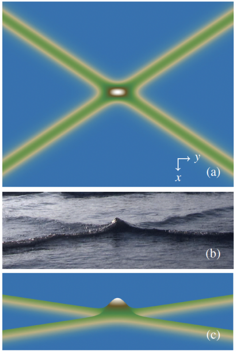
$$u_{[3,4]}(X, Y, \tau) = \frac{1}{2}(k_4 - k_3)^2 \mathrm{sech}^2 \frac{1}{2}(\theta_3 - \theta_4 - \ln b),$$

where a, b depend on k_j. For equal far-field soliton amplitudes $\tilde{A} = \frac{1}{2}(k_2 - k_1)^2 = \frac{1}{2}(k_4 - k_3)^2$, the solution satisfies [Kodama, 2010]

$$2\tilde{A} \leq \max_{(X,Y,\tau)} u(X, Y, \tau) \leq 2\left(1 + \frac{1 - \sqrt{\Delta_o}}{1 + \sqrt{\Delta_o}}\right)\tilde{A},$$

where $0 \leq \Delta_o \leq 1$, hence $\boxed{2\tilde{A} \leq \max u \leq 4\tilde{A}}$.



Example: three interacting line solitons

Three line solitons, known as $(3,3)$-solitons, have $(N,M)=(3,6)$ and functions $f_1=\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}$, $f_2=\mathrm{e}^{\theta_3}+\mathrm{e}^{\theta_4}$, $f_3=\mathrm{e}^{\theta_5}+\mathrm{e}^{\theta_6}$, and

$$K(X,Y,\tau)=\underline{A_{135}}\,\mathrm{e}^{\theta_1+\theta_3+\theta_5}+\underline{\underline{A_{235}}}\,\mathrm{e}^{\theta_2+\theta_3+\theta_5}+\underbrace{A_{136}}\,\mathrm{e}^{\theta_1+\theta_3+\theta_6}+A_{236}\,\mathrm{e}^{\theta_2+\theta_3+\theta_6}$$
$$+A_{145}\,\mathrm{e}^{\theta_1+\theta_4+\theta_5}+\underline{\underline{A_{245}}}\,\mathrm{e}^{\theta_2+\theta_4+\theta_5}+\underbrace{A_{146}}\,\mathrm{e}^{\theta_1+\theta_4+\theta_6}+\underline{A_{246}}\,\mathrm{e}^{\theta_2+\theta_4+\theta_6},$$

with the following parameter ordering $k_1<k_2<k_3<0<k_4<k_5<k_6$.

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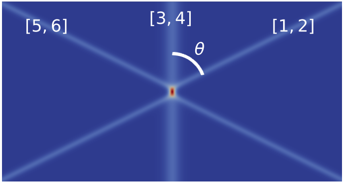
$$+ A_{145}\,\mathrm{e}^{\theta_1+\theta_4+\theta_5} + \underline{\underline{A_{245}}}\,\mathrm{e}^{\theta_2+\theta_4+\theta_5} + \underbrace{A_{146}}\,\mathrm{e}^{\theta_1+\theta_4+\theta_6} + \underline{A_{246}}\,\mathrm{e}^{\theta_2+\theta_4+\theta_6},$$

with the following parameter ordering $k_1 < k_2 < k_3 < 0 < k_4 < k_5 < k_6$.

In the far field $Y \to \pm\infty$, we find the single line solitons

$$u_{[1,2]} \approx \frac{1}{2}(k_2-k_1)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1-\theta_2-\ln\tilde{a}),$$
$$u_{[5,6]} \approx \frac{1}{2}(k_6-k_5)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_5-\theta_6-\ln\tilde{b}),$$
$$u_{[3,4]} \approx \frac{1}{2}(k_4-k_3)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_3-\theta_4),$$



with $\theta_i - \theta_j = (k_i - k_j)\Big(X + (k_i+k_j)Y - (k_i^2 + k_i k_j + k_j^2)\tau\Big)$.

Example: three interacting line solitons

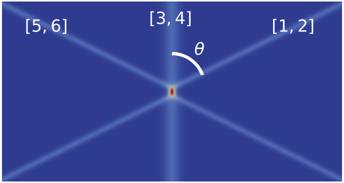
Parameters $k_1, \ldots, k_6$ are determined from

$$k_3 + k_4 = 0$$
$$k_5 + k_6 = -(k_1 + k_2) = \tan\theta$$
$$k_4 - k_3 = \sqrt{2\tilde{A}}$$
$$k_6 - k_5 = k_2 - k_1 = \sqrt{2\tilde{A}/\lambda}$$

where angle $\theta > 0$, $\tilde{A} = \frac{1}{2}(k_4 - k_3)^2$ is the amplitude of the $[3,4]$ soliton, and the outer two solitons are assumed to have amplitude $\tilde{A}/\lambda$, for $\lambda \geq 1$.

Solving the above six equations, gives

$$k_6 = -k_1 = \sqrt{\tilde{A}}\left(\sqrt{2/\lambda} + \sqrt{1/2} + \delta\right)$$
$$k_5 = -k_2 = \sqrt{\tilde{A}}\left(\sqrt{1/2} + \delta\right)$$
$$k_4 = -k_3 = \sqrt{\tilde{A}/2}$$

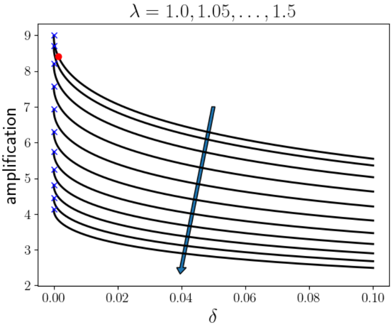


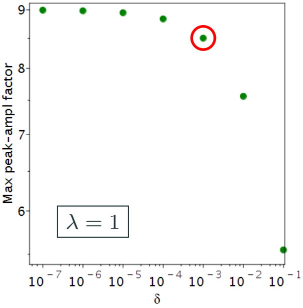
where δ is defined by

$$\delta = \frac{\tan\theta}{2\sqrt{\tilde{A}}} - \left(\sqrt{1/2\lambda} + \sqrt{1/2}\right) > 0.$$

Proof of maximum amplification

- Proof is based on a geometric argument (additional secondary proof)
- Find centreline of each of three line solitons (no phase shift at peak)
- Look for intersection points $\rightsquigarrow$ this gives two values of Y, with mean at a unique point Y_* when $\tau_* = 0$ and $X_* = 0$
- The space-time point of maximum amplification is (X_*, Y_*, τ_*)
- Amplification: $\dfrac{u(X_*, Y_*, \tau_*)}{\tilde{A}} = \dfrac{(\sqrt{\lambda}+2)^2}{\lambda} + \mathcal{O}(\sqrt{\delta}) \xrightarrow[\delta=0]{} 1 + \dfrac{4}{\lambda} + \dfrac{4}{\sqrt{\lambda}}$





Numerical implementation

Firedrake — *An automated system for the solution of PDEs using the Finite Element Method (FEM).*

Firedrake employs Unified Form Language (UFL) and linear & non-linear solvers PETSc solvers [Rathgeber et al., 2016].

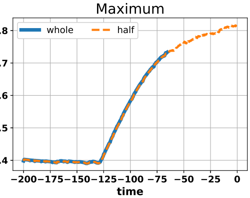
- Space-time discretisation 2nd order of variational principle for BL: bounded energy oscillations, phase-space conserved.
- Continuous Galerkin (CG) FEM in space, with approximations and test functions w_k given by variations $\delta\eta_h,\ \delta\Phi_h$:

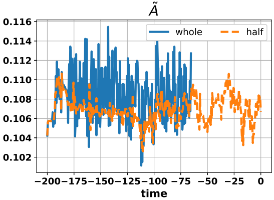
$$\eta(x,y,t) \approx \eta_h(x,y,t) = \sum_k \eta_k(t) w_k(x,y),\ \dots$$

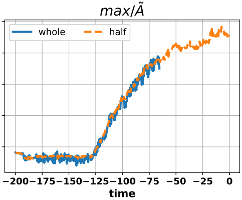
- Symplectic Störmer-Verlet time stepping scheme.
- Stable numerical scheme: no artificial amplitude damping ...

Computational domain: $\sim$cnoidal waves

- KP solutions hold on infinite horizontal plane, so domain has to be sufficiently large to eliminate reflection at boundaries.
- Solutions can be set to become approximately periodic in sufficiently large domains.
- Transform $\Phi = (U_0 x + c_0) + \tilde{\Phi}$, where $\tilde{\Phi}$ is periodic, then solve the Benney-Luke equations for η and $\tilde{\Phi}$.
- Doubly or singly periodic domain?







Initial conditions and boundaries

Initial condition consists of two (SP2) or three (SP3) line solitons, expressions of which are known from the KP-solution:

$$\eta_0(x,y) = \eta(x,y,t_0) = 2\left(\frac{4}{3}\right)^{1/3}\partial_{XX}\ln K(X,Y,\tau),$$

$$\Phi_0(x,y) = \Phi(x,y,t_0) = 2\sqrt{\epsilon}\left(\frac{4\sqrt{2}}{9}\right)^{1/3}\partial_X \ln K(X,Y,\tau).$$

Computational domain is constructed such that initial condition satisfies "periodic boundary conditions" in x–direction.

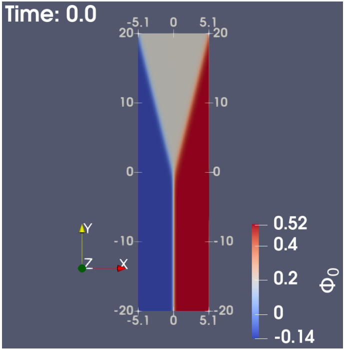
Case	L_x	L_y	T	N_x	N_y	$\Delta x = \frac{L_x}{N_x}$	$\Delta y = \frac{L_y}{N_y}$	Δt
SP2	10.3	40	50	132	480	0.0779	0.0833	0.005
SP3	20.9	47	200	252	564	0.0829	0.0833	0.005

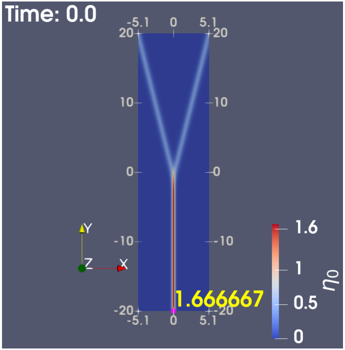
Overview of simulations

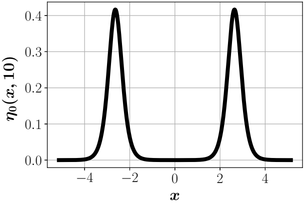
- SP2: two-soliton interaction, with $\tilde{A} \approx 0.4$, $\delta = 0$
- SP3: three-soliton interaction, with $\tilde{A} \approx 0.1$, $\lambda = 1$, $\delta \approx 0.001$
- Higher-order polynomial resolution CG2/CG3 for convergence
- Benney-Luke numerical solutions consistent with exact KP solutions, difference within order ϵ (here, $\epsilon = 0.05$)

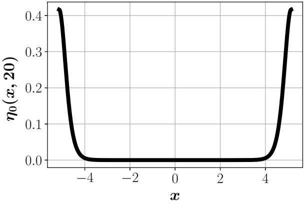
Case	Equation	$\tilde{A}$	$\max$	$\max / \tilde{A}$
	η_{KP}	~ 0.415	~ 1.66	4.0
SP2	η	$0.399 - 0.435$	$1.56 - 1.69$	$3.64 - 4.01$
	η_{KP}	~ 0.104	$0.396 - 0.875$	8.41
SP3	η	$0.103 - 0.111$	$0.390 - 0.818$	$3.60 - 7.83$

Results for two-soliton interaction









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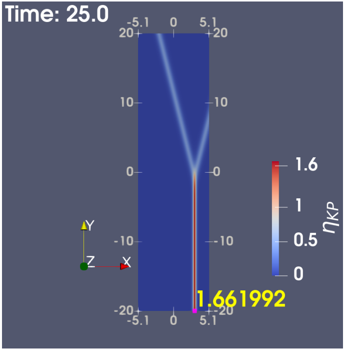
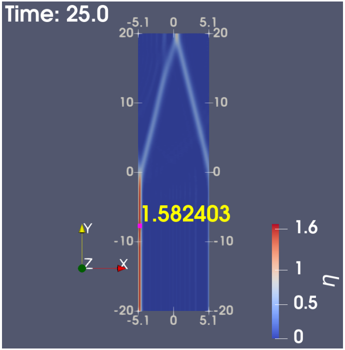
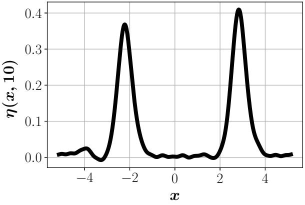
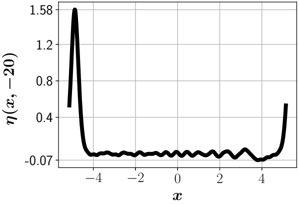
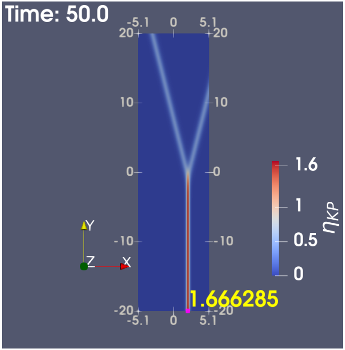


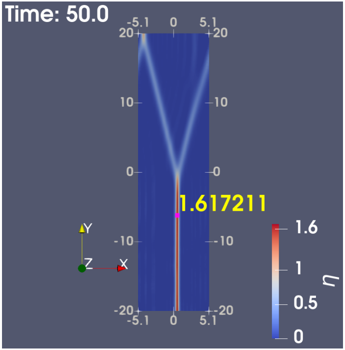


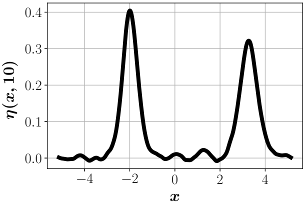


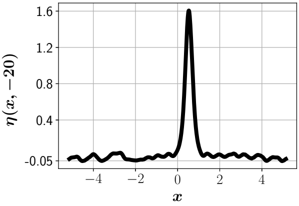


Results for two-soliton interaction









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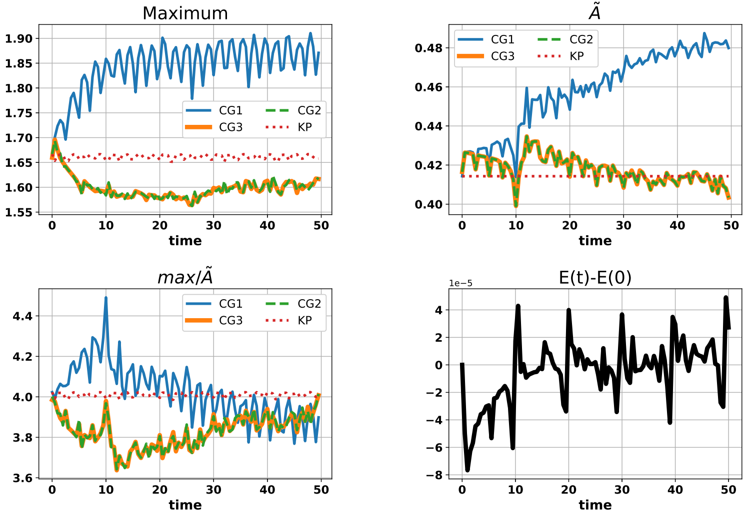

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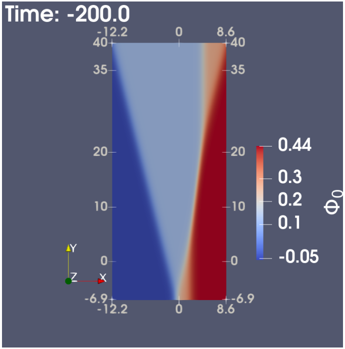

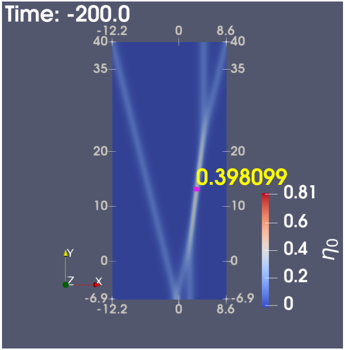

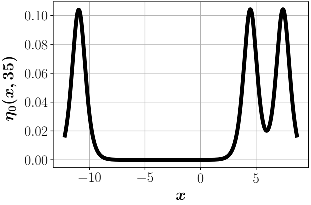

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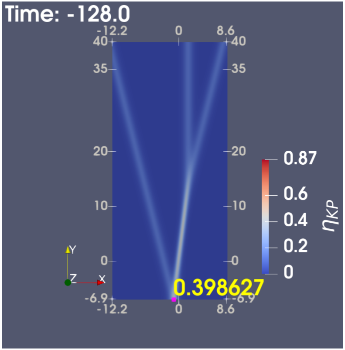

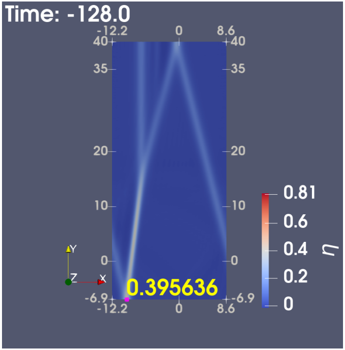

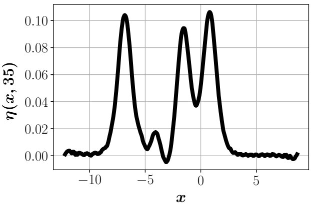

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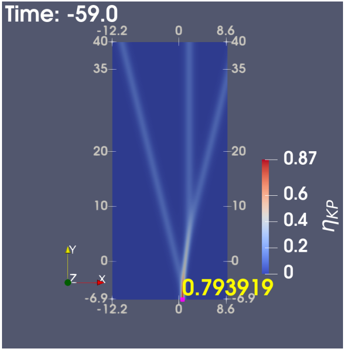
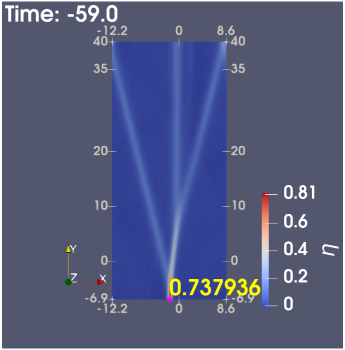
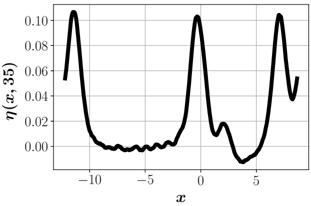
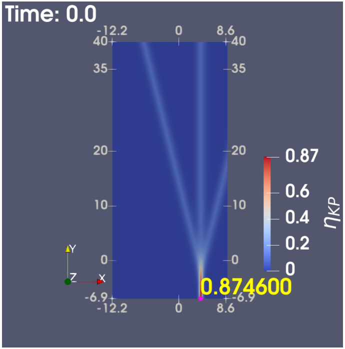
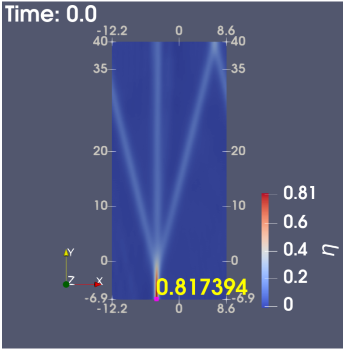


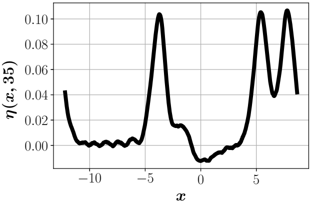




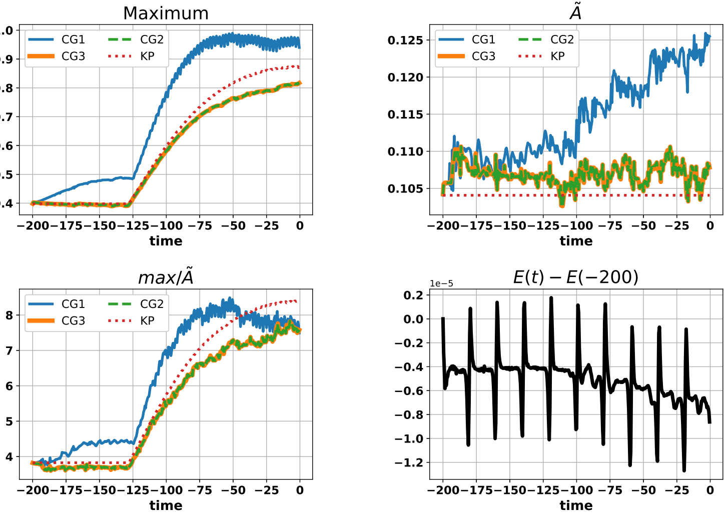
Results for three-soliton interaction





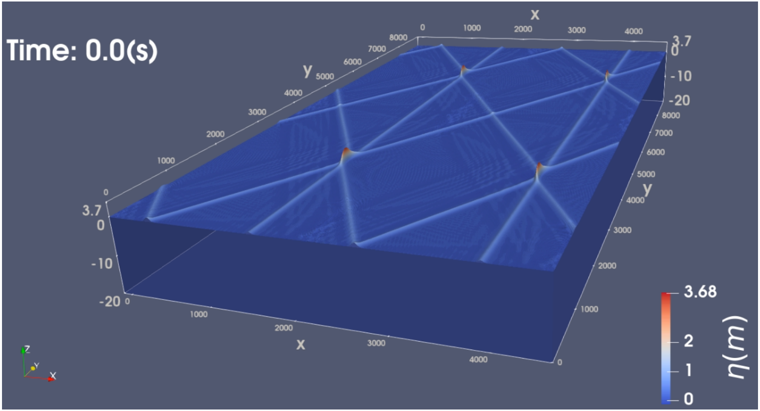


Results for three-soliton interaction



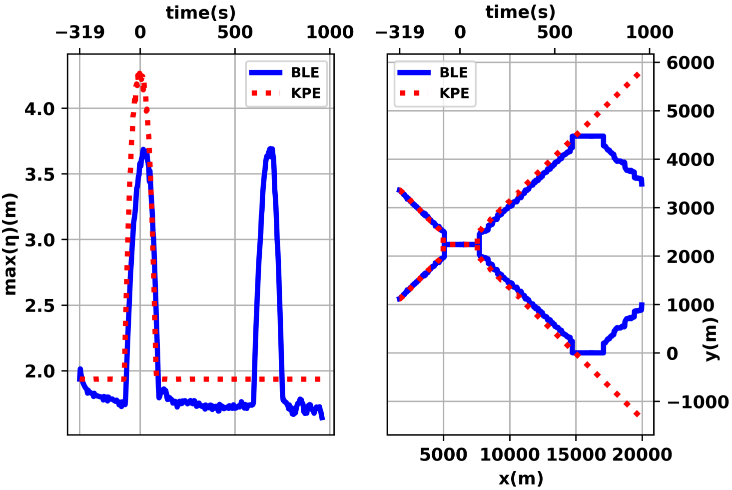
Results for three-soliton interaction (dimensional)

Crossing seas (four/eight domains combined)

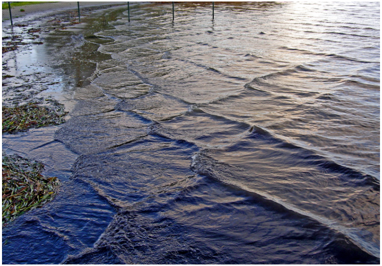


Results for three-soliton interaction (dimensional)

Cnoidal waves with periodicity in x, y, t (max. vs. t & x–y tracks):



Summary



- Nine-fold soliton amplification shown theoretically, but only in limit $\delta \to 0$
- Web-soliton amplification KP≈ 8.4, simulated for BL≈ 7.8
- Amplifications achieved as simulated cnoidal crossing seas
- Rogue wave calculation: $\mathrm{AI} > 2$ to $4.0 = \max(\eta)/H_s$
- Can we reach nine-fold amplification in BL?
- Can amplifications survive in potential-flow modelling/lab?

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- Crossing seas *YouTube movie*

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