Treatment of Density Variations in Ice-Flow Models using the Shallow Ice Stream Approximation

Camilla Schelpe* & Hilmar Gudmundsson

Northumbria University, Newcastle-Upon-Tyne, UK
**camilla.schelpe@northumbria.ac.uk*

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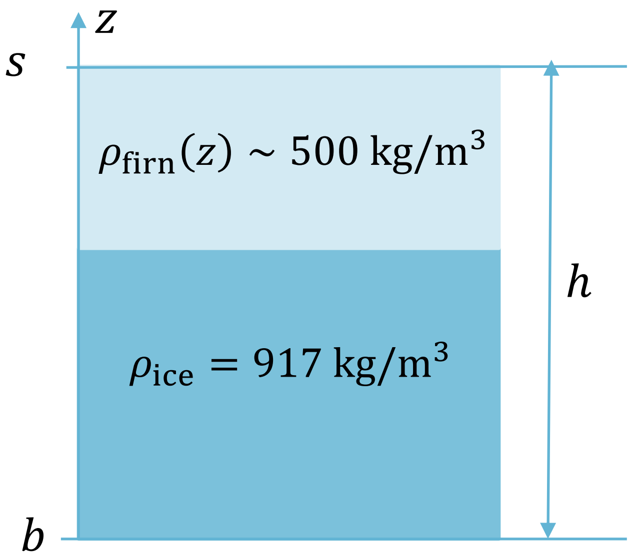
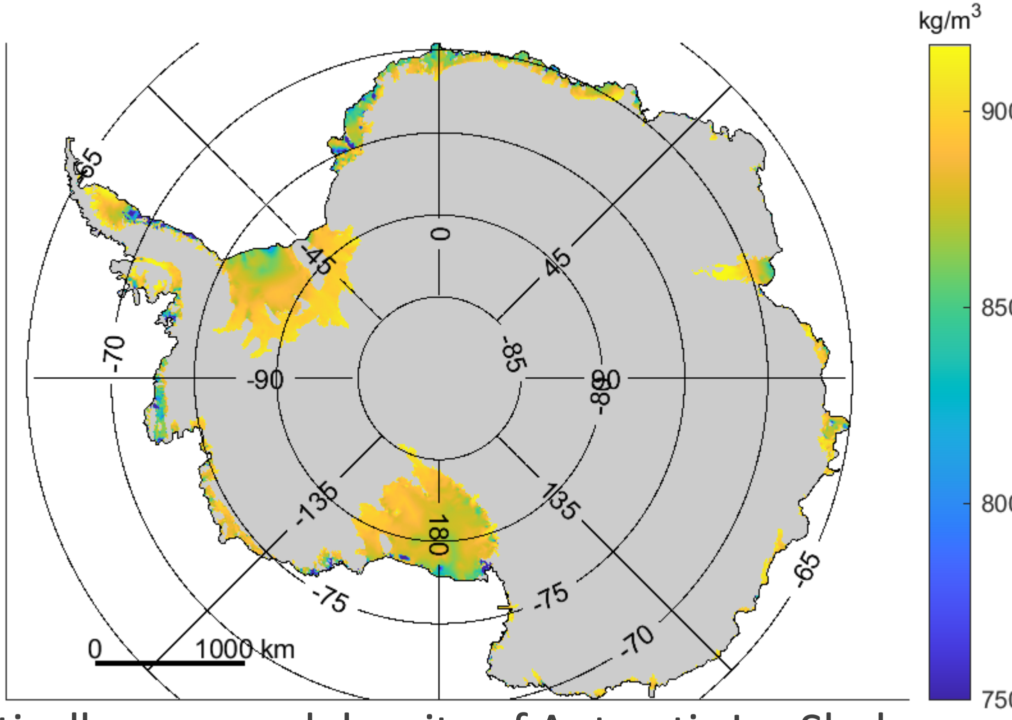
Density Variations in Ice Sheets

Ice Sheet = Ice layer + Firn layer



Vertically-averaged density,

$$\langle \rho \rangle (x,y) = \frac{1}{h}\int_b^s \rho(z)\, \text{dz}$$



Vertically-averaged density of Antarctic Ice Shelves (BedMachine Antarctica ©*Morlighem et al. 2020*):

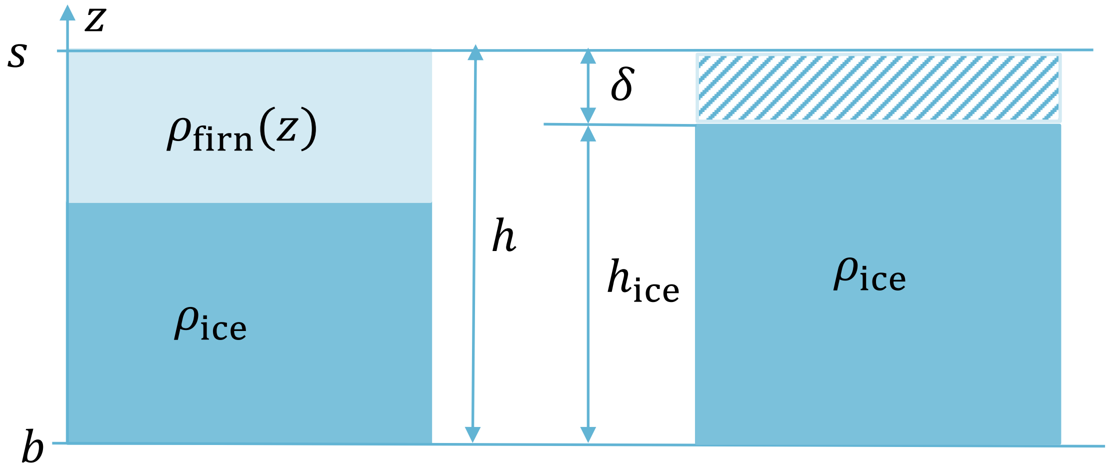
$\langle \boldsymbol{\rho} \rangle \sim \mathbf{917\ - 790\ kg/m^3},$

more than 10% spatial variation in density.

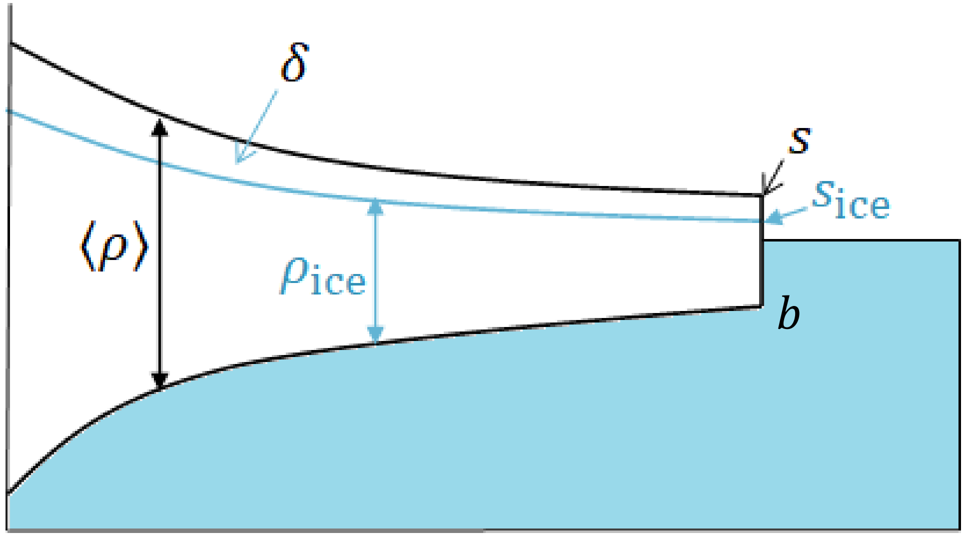
Density-to-Thickness Adjustment [D2T]

Firn air-content: $\delta \equiv h\left(1 - \frac{\langle\rho\rangle}{\rho_{\text{ice}}}\right)$

D2T adjustment: set $\rho = \rho_{\text{ice}}$ and adjust the thickness of the ice sheet: $h_{\text{ice}} = h - \delta$.



e.g. Floating Ice Shelf

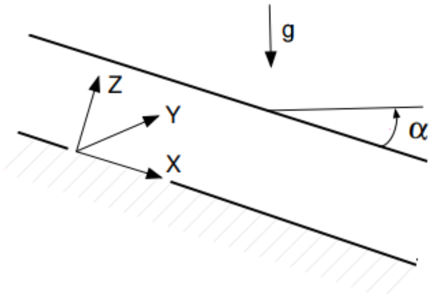


$$\langle\rho\rangle \times h = \rho_{\text{ice}} \times h_{\text{ice}}$$

The SSA Equations with Horizontal Density Variations

Vertically-integrated SSA momentum-conservation:

- $\partial_x \left(4h\eta\partial_x u + 2h\eta\partial_y v + \frac{2h\eta}{\rho}\frac{D\rho}{Dt}\right) + \partial_y \left(h\eta(\partial_x v + \partial_y u)\right) - t_{bx} = \rho g h(\partial_x s \cos\alpha - \sin\alpha) + \frac{1}{2} g h^2 \partial_x \rho \cos\alpha$
- $\partial_y \left(4h\eta\partial_y v + 2h\eta\partial_x u + \frac{2h\eta}{\rho}\frac{D\rho}{Dt}\right) + \partial_x \left(h\eta(\partial_x v + \partial_y u)\right) - t_{by} = \rho g h\partial_y s \cos\alpha + \frac{1}{2} g h^2 \partial_y \rho \cos\alpha$



Mass-conservation:

$$\frac{D\rho}{Dt} + \rho\, \nabla \cdot \mathbf{v} = 0$$

- On the **LHS**: additional momentum transfer between regions of low and high density.
- On the **RHS**: additional gravitational driving term due to horizontal density gradient.
- Both terms contribute equally $\sim\Delta\rho/\rho \lesssim 10\%$. However the RHS correction may dominate for grounded ice-caps, and also ice-streams/ice-shelves where $\partial_x s \approx 0$.

Horizontal Density Variation Approaches

Typical ice-flow models

Density-to-Thickness adjustment [D2T]:

Set $\rho = \rho_{\text{ice}}$ and adjust the thickness of the ice sheet by the firn air-content: $h_{\text{ice}} = h - \delta$.

+ no modification to SSA eqns required

– modifies all terms involving h in the SSA eqns, not just density gradients

Density Variations - Body Force Only [DV-BF]:

Úa

Model a static density distribution $\rho(x, y)$ but neglect correction term on LHS of mom. eqn.

+ more realistic model of density variations

– may not be accurate in scenarios with significant longitudinal stretching and steep gradients

Density Variations Advected [DVA]:

Allow density to evolve, e.g. set $\mathrm{D}\rho/Dt = 0$ such that $\partial_t \rho = -\mathbf{v} \cdot \boldsymbol{\nabla}\rho$.

+ more flexible formulation

– requires snow accumulation and compacitification model to be realistic

Density Variations [DV]:

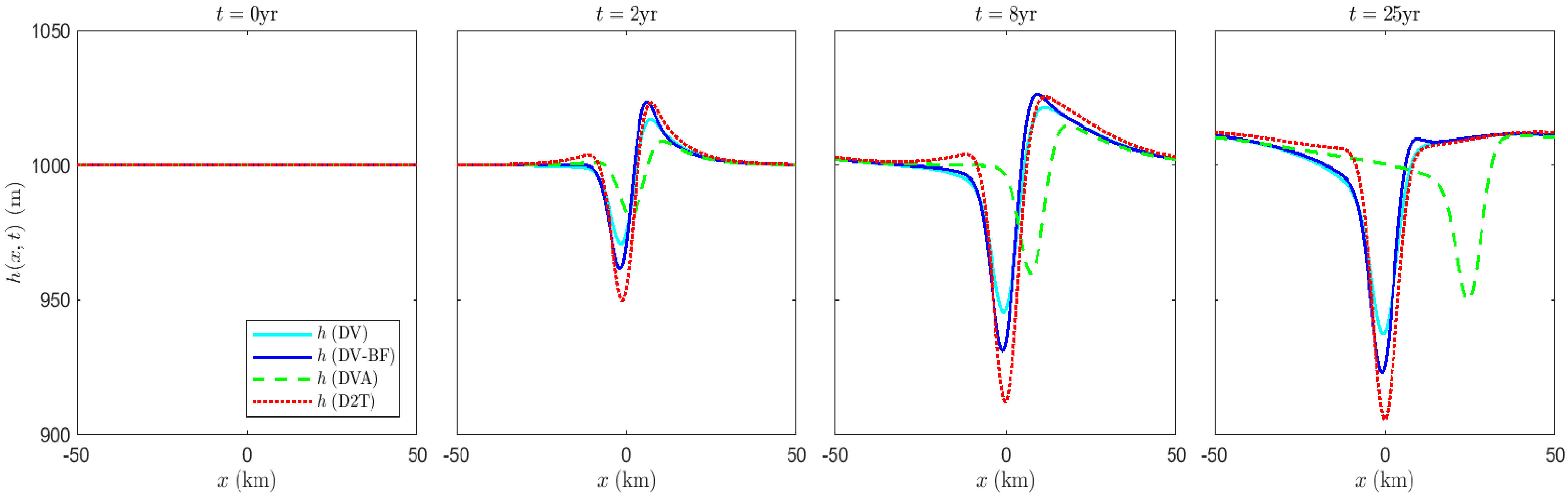
Model a static density distribution $\rho(x, y)$ including both correction terms, i.e. $\mathrm{D}\rho/Dt = \mathbf{v} \cdot \boldsymbol{\nabla}\rho$.

+ more realistic model of density variations

– momentum eqn is no longer frame-invariant and so difficult to implement in large-scale flow models

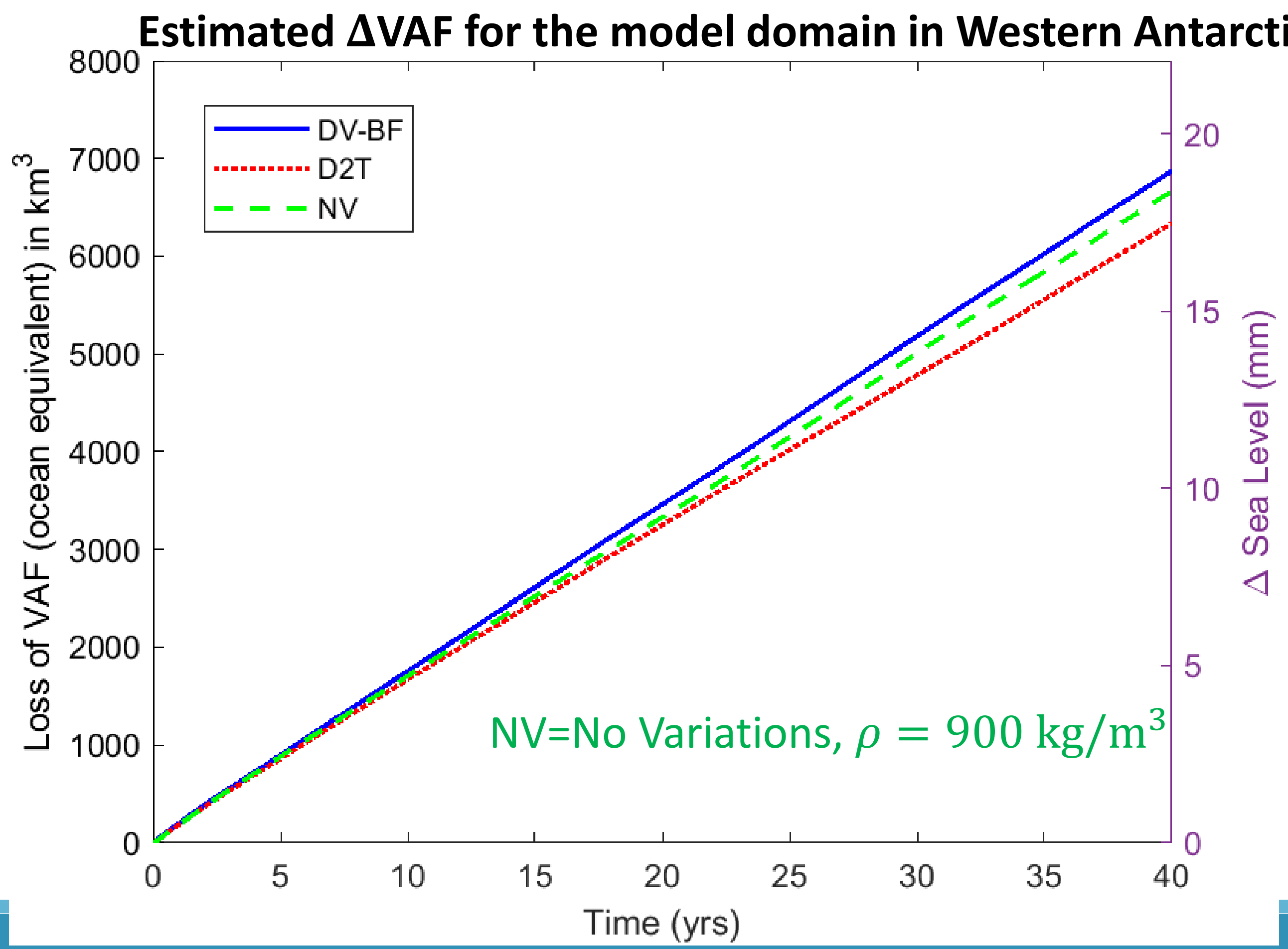
Analytical Simulation: Grounded Ice Sheet Example

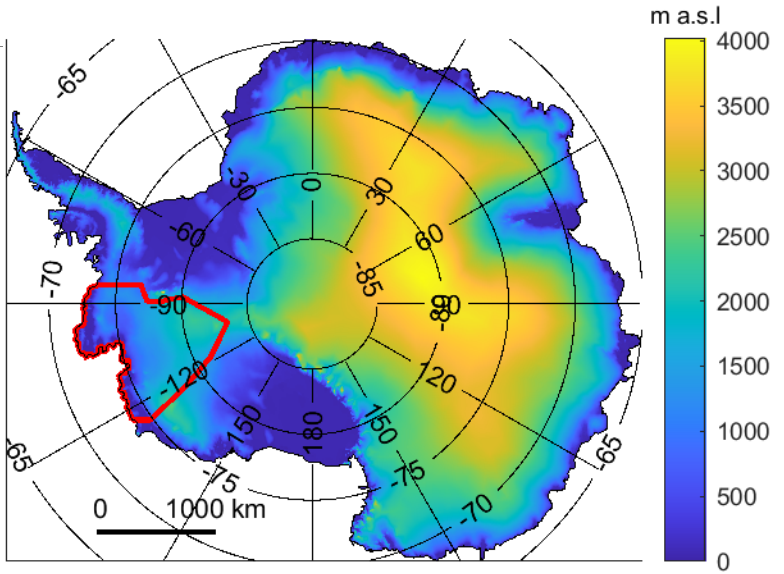
infinite horizontal span, at an inclined slope α

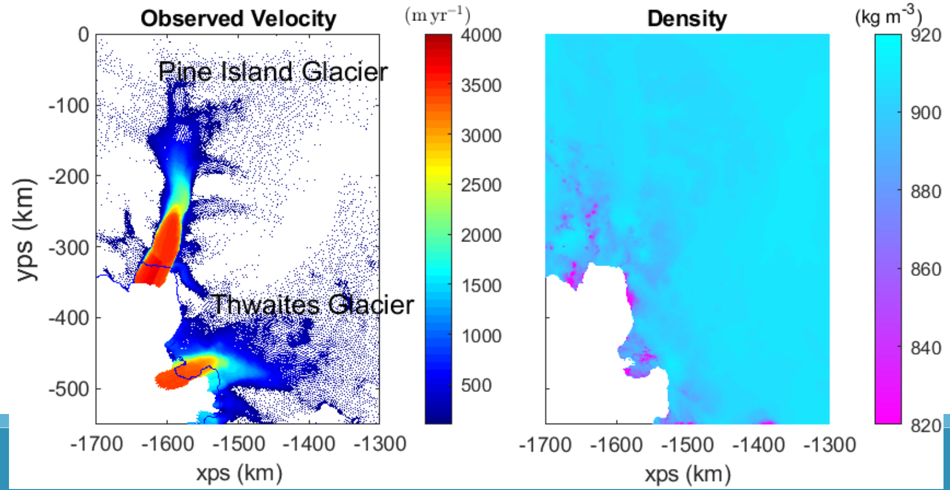


Surface perturbation in response to a small Gaussian density perturbation applied to the ice sheet

Numerical Simulation of Western Antarctica







Conclusions

- We have derived a new theoretical framework for the inclusion of Horizontal Density Variations in large-scale ice-flow models.
- Different formulations are proposed for handling Horizontal Density Variations: D2T, DV-BF, DV, DVA.
- A ball-park estimate expects approximately a 10% correction due to observed density variations.
- Perturbation Analysis (to first order in the density variations) reveals a number of qualitative and quantitative differences between different density formulations.
- Transient simulations of the Western Antarctic Ice Sheet suggest that the impact on predicted sea level change could be approximately 10% between different formulations.