Are avalanche models correct?
An uncertain view on convergence

Wildbach- und Lawinenverbauung Forsttechnischer Dienst

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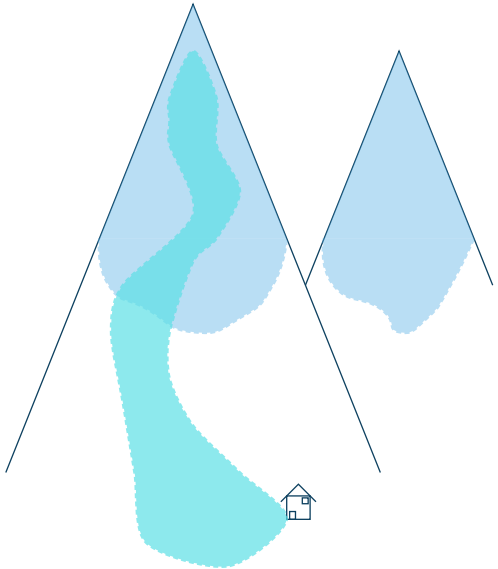
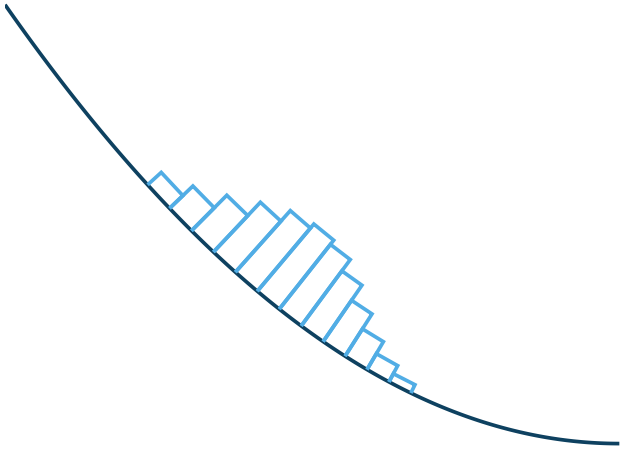
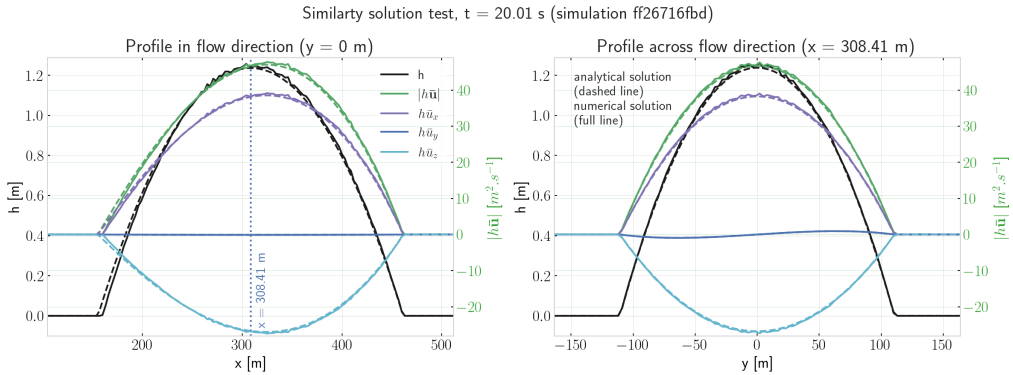


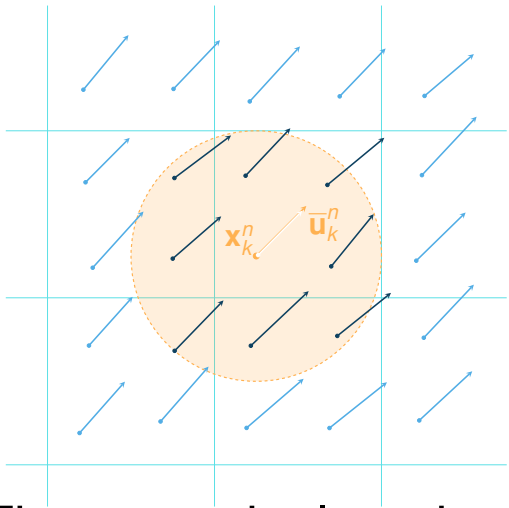
The real world



The physical model



Validating the model



The numerical method

Read the abstract · · ·

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At the core of many avalanche simulation tools, numerical kernels are utilized to solve flow model equations. Aside from trying to fit the models as best as possible to the current understanding of actual flow mechanisms, these kernels have to fulfill general mathematical requirements, such as convergence, stability and consistency. The precision of numerical solutions is limited and needs to be determined by appropriate uncertainty quantification approaches. It is also necessary to assess the impact of input variability propagating through the numerical kernel.
To allow kernel testing and uncertainty quantification, the AvaFrame framework provides a suite of test cases as well as analysis tools. This includes tests with known solutions usable to determine the kernel errors (ana1Tests) and idealized/real world topographies to estimate effects of varying simulation setups. By changing numerical settings, flow model setup or input data it is possible to show their effects on simulation results in a quantitative manner. It therefore allows us to relate input variations to the uncertainty in simulation results. Error and uncertainty quantification is done using modules for computing statistical measures (ana4Stats), indicators along an avalanche path (ana3AIMEC) and various visualization routines.
We showcase this for our com1DFA dynamical dense flow avalanche (DFA) module. The kernel of com1DFA is based on depth integrated governing equations (shallow water) and solved numerically using the smoothed particle hydrodynamics (SPH) method. Applying our analysis tools, we evaluate the convergence of the DFA kernel with regard to the numerical parameters time step, SPH kernel size and particles size. We investigate the accuracy and precision of the numerical solution using the similarity solution test, a test with a semi-analytic solution for depth integrated equations. It allows us to establish a suitable relation between time step, SPH kernel size and particles size for the com1DFA kernel.
Using the same approach for an avalanche setup, we can also vary selected input parameters like friction coefficients and/or release thickness and quantify the resulting uncertainties on simulation results, e.g. runout and peak flow variables.

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From 3D N-S to thickness averaged equations

- Navier-Stokes (N-S) equations
- Shallow flow for an incompressible material
- Neglecting higher order terms (lateral shear stresses) and assuming mild curvature (compared to flow thickness)

In terms of particular derivative (Lagrangian approach)

In 3D

$$\begin{cases} \frac{\mathrm{d}\rho_0}{\mathrm{d}t} = 0 \\ \frac{\mathrm{d}\rho_0 \mathbf{u}}{\mathrm{d}t} = -\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \rho_0 \mathbf{g} \\ \text{Boundary conditions} \end{cases}$$

After thickness integration

$$\begin{cases} \frac{\mathrm{d}\rho_0 A h}{\mathrm{d}t} = 0 \\ \rho_0 A h \frac{\mathrm{d}\bar{\mathbf{u}}}{\mathrm{d}t} = -\rho_0 A h g_n \boldsymbol{\nabla} h \\ \qquad\qquad -A\boldsymbol{\tau}^{\mathrm{bot}} + \rho_0 A h \mathbf{g} \end{cases}$$

Now discretizing the equations: **Numerics**

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A grid-particle numerical method

$$\frac{\mathrm{d}\rho_0 A h}{\mathrm{d}t} = 0, \quad \rho_0 A h \frac{\mathrm{d}\bar{\mathbf{u}}}{\mathrm{d}t} = -\rho_0 A h g_n \nabla h - A \tau^{\mathrm{bot}} + \rho_0 A h \mathbf{g}$$

Mass discretization : Particle method

- Mass is tracked by particles (mass $m_k = \rho_0 A_k h_k$)
- Momentum equation solved for each particle $m_k \bar{\mathbf{u}}_k$
- Computing flow thickness gradient ∇h using **SPH**

$$\frac{\mathrm{d}m_k}{\mathrm{d}t} = 0, \quad m_k \frac{\mathrm{d}\bar{\mathbf{u}}_k}{\mathrm{d}t} = -m_k g_n \nabla h_k - A_k \tau_k^{\mathrm{bot}} + m_k \mathbf{g}$$

Particle to grid interpolation

m and $\bar{\mathbf{u}}$ interpolated onto grid using bilinear interpolation

combined with the following time discretization $\cdots$

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Time discretization : Euler time scheme + operator splitting

- Add driving force: $\bar{\mathbf{u}}_k^{n+1\,*} = \bar{\mathbf{u}}_k^n + \frac{\mathbf{F}_\mathbf{k}^{\text{drive}} \Delta t}{m}$
- Add friction force: $\bar{\mathbf{u}}_k^{n+1} = \bar{\mathbf{u}}_k^{n+1\,*} + \frac{\mathbf{F}_\mathbf{k}^{\text{fric}} \Delta t}{m}$

  only if $\bar{\mathbf{u}}_k^{n+1*} > \frac{\mathbf{F}_\mathbf{k}^{\text{fric}} \Delta t}{m}$, otherwise particle stops. This prevents the friction force to become a driving force and allows proper and physical stopping
- Update position: $\mathbf{x}_k^{n+1} = \mathbf{x}_k^n + \frac{1}{2}\left(\bar{\mathbf{u}}_k^{n+1} + \bar{\mathbf{u}}_k^n\right)\Delta t$

Now discretizing the flow thickness gradient

using an SPH method · · ·

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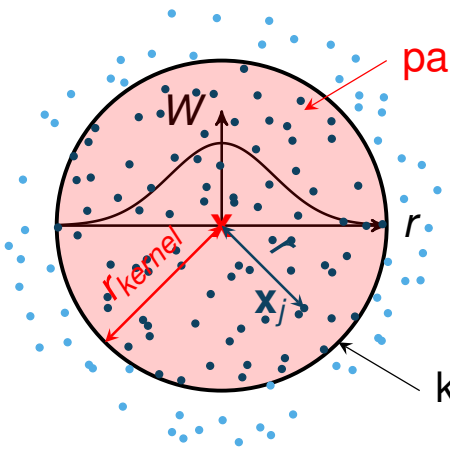
Smoothed Particle Hydrodynamics (SPH)

An interpolation method in the first place

- $\langle f(\mathbf{x}) \rangle = \sum_j f(\mathbf{x}_j) W(\mathbf{x} - \mathbf{x}_j) A_j$
- With its specific characteristics (not exact at node locations...)
- $\langle \nabla f(\mathbf{x}) \rangle = -\sum_j f(\mathbf{x}_j) \nabla W(\mathbf{x} - \mathbf{x}_j) A_j$



particle neighborhood Ω

kernel function support $\partial\Omega$

Which gives for $\nabla h \cdots$

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Computing flow thickness gradient using SPH

For each particle:

$$\text{mass} \quad m_j = \rho_0 A_j h_j$$

Expressing the flow thickness gradient using SPH:

$$\langle \nabla h(\mathbf{x}) \rangle = -\sum_j h(\mathbf{x}_j) \nabla W(\mathbf{x} - \mathbf{x}_j) A_j = -\sum_j \frac{m_j}{\rho_0} \nabla W_{ij}$$

Flow thickness is not needed to compute its gradient!!

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Towards convergence

Question: does the numerical solution converge towards the analytical solution of the depth integrated equation?

Discretization parameters influencing the convergence:

- time step dt (+ time discretization method)
- SPH kernel radius (influences spatial resolution of the gradient computation)
- number of particles (influences accuracy of the gradient computation)

more on these parameters · · ·

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More details about the parametrization

The number of particles is not defined by a particle size but by n_{ppk}, the number of particles within a circle of radius r_{kernel}:

$$n_{\text{ppk}} = C_{\text{ppk}} r_{\text{kernel}}^{\alpha} = n_{\text{ppk}}^{0} \left(\frac{r_{\text{kernel}}}{r_{\text{kernel}}^{0}} \right)^{\alpha}$$

where n_{ppk}^{0} is the number of particles within a circle of reference radius r_{kernel}^{0}. This leads to the following expression for the particle size:

$$r_{\text{part}} = \left(\frac{{r_{\text{kernel}}^{0}}^{\alpha}}{n_{\text{ppk}}^{0}} \right)^{1/2} r_{\text{kernel}}^{1-\alpha/2}$$

This brings us to the convergence criterion $\cdots$

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▲ Convergence criterion by Ben Moussa & Vila:

$$\begin{cases} r_{part} \to 0 \\ r_{kernel} \to 0 \\ \dfrac{r_{part}^{m}}{r_{kernel}^{m+1}} \to 0 \quad m = 2 \end{cases} \quad \text{and} \quad dt \leq C r_{kernel}$$

In other words:

$$\begin{cases} dt = C_{time} r_{kernel} \\ n_{ppk} = n_{ppk}^{0} \left(\dfrac{r_{kernel}}{r_{kernel}^{0}} \right)^{\alpha} \end{cases}$$

Convergence expected for (α is called aPPK in the following figures):

$$\boxed{\alpha < -1}$$

Moussa, B. and Vila, J. P.: Convergence of SPH Method for Scalar Nonlinear Conservation Laws, Siam Journal on Numerical Analysis - SIAM J NUMER ANAL, 37, https://doi.org/10.1137/S0036142996307119, 2000.

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The implementation is validated on tests with known solutions. Then the runout of a real case avalanche is studied for varying release thickness.

Testing

- Error measurement method
- Similarity solution test
- Dam break test
- Results

Application

- Uncertainty propagation

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How to measure the error?

Error on flow thickness h and flow momentum $|h\bar{\mathbf{u}}|$

$$\epsilon(x) = f_{\text{num}}(x) - f_{\text{ref}}(x)$$

- Largest absolute value of the deviation $\epsilon(x)$ on Ω:

$$\mathcal{L}_{\max}(\epsilon(x)) = \sup_{x\in\Omega}(|\epsilon(x)|) \quad \text{and} \quad \frac{\mathcal{L}_{\max}(\epsilon(x))}{\mathcal{L}_{\max}(f_{\text{ref}}(x))}$$

- Euclidian error:

$$\mathcal{L}_2(\epsilon(x)) = \int_{x\in\Omega} ||\epsilon(x)||^2 \, dx \quad \text{and} \quad \frac{\mathcal{L}_2(\epsilon(x))}{\mathcal{L}_2(f_{\text{ref}}(x))}$$

The similarity solution test case · · ·

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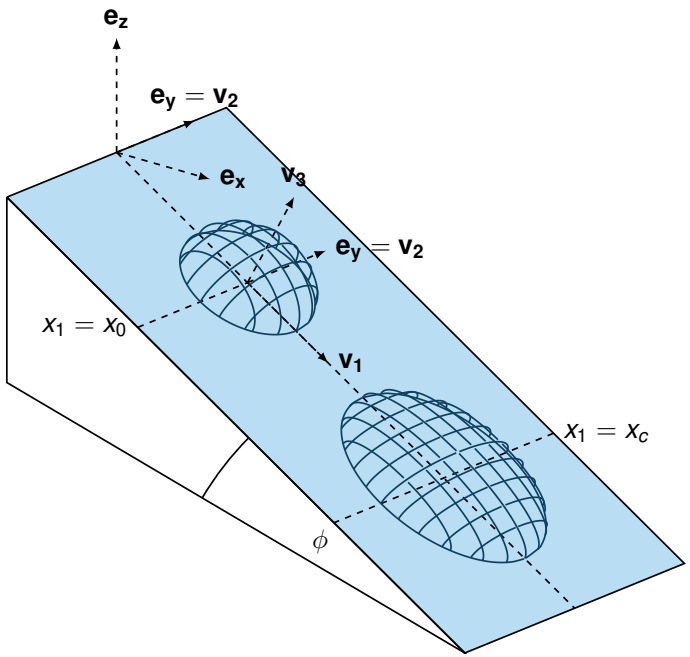
Smilarity solution test



- Half ellipsoidal shape
- Flowing down an inclined plane of slope ϕ
- Coulomb friction $A\,\tau_{bot} = -\mu\, m\, g_3\, \frac{\bar{\mathbf{u}}}{||\bar{\mathbf{u}}||}$

Figure: 3D view of the material on the inclined plane at $t = 0$ and at a general time t for the similarity solution test case

Similarily we have the dam break test

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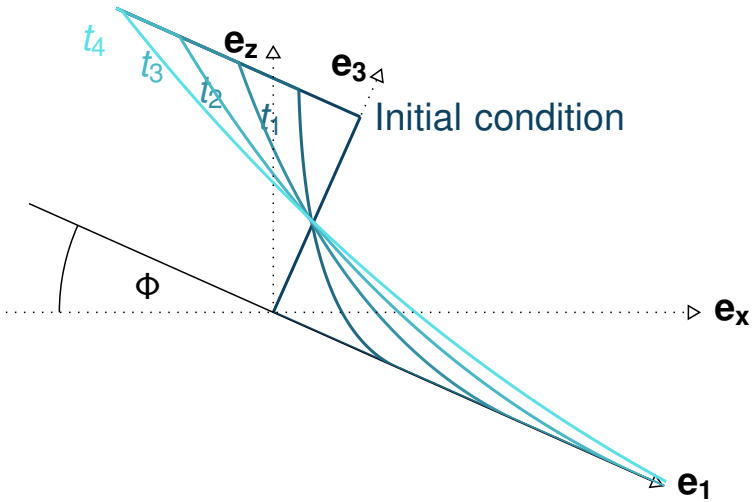
Dam break test



- Material suddenly released from rest
- Flowing down an inclined plane of slope ϕ
- Coulomb friction

$$A\,\boldsymbol{\tau_{bot}} = -\mu\, m\, g_3\, \frac{\bar{\mathbf{u}}}{||\bar{\mathbf{u}}||}$$

$$(h, \mathbf{u})(x, t=0) = \begin{cases} (h_0, \mathbf{0}), & \text{if } x \leq 0 \\ (0\ \ , \mathbf{0}), & \text{if } x > 0 \end{cases}$$

Figure: 2D view of the material on the inclined plane for the dam break test case at $t = 0$ and for different $t > 0$

Varying the discretization parameters leads to the following results

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Numerical parameter variation

For both tests, simulations are computed varying the kernel radius size (sphKernelRadius), the α exponent (aPPK), the time step via C_{time} (cMax) and the reference number of particles per kernel radius (nPPK0).
The reference kernel radius (sphKR0) is kept fixed.
For each variation, the error between the simulation and analytical result is computed.
The following figure shows how these parameters influence the convergence of the solution on the dam break test.
This approch applied to the similarity solution case shows similar results.

Convergence of the dam break test · · ·

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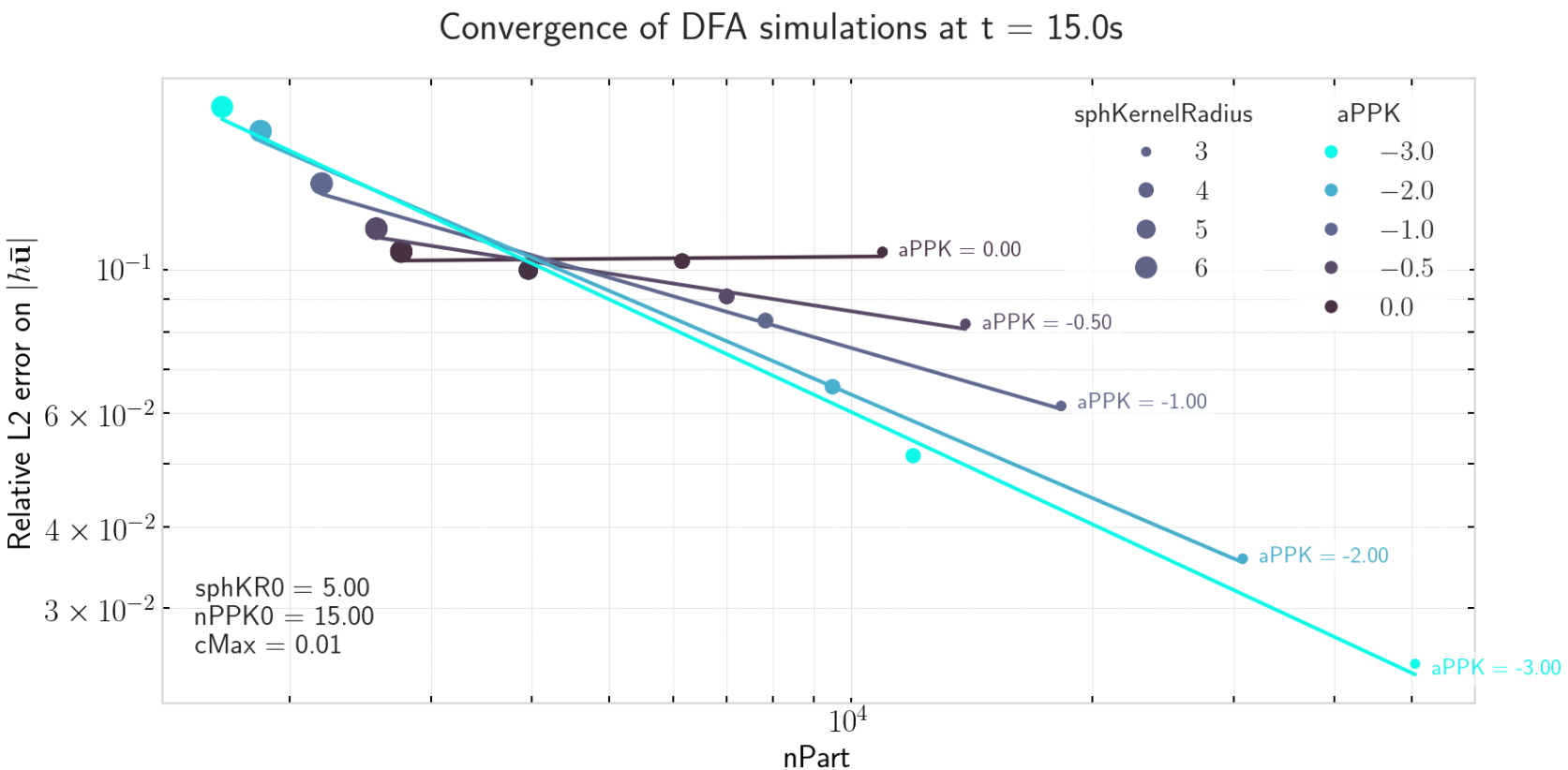
Convergence of dam break test



Which brings us to the conclusion · · ·

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Convergence analysis

- The smaller the α (aPPK) exponent, the faster the solution seems to converge
- For exponents bigger then -1, the convergence is slow or not observed
- Decreasing C_{time} (called cMax in the figures) constant or increasing the particle density (via nPPK0) leads to smaller error (not shown on figures)

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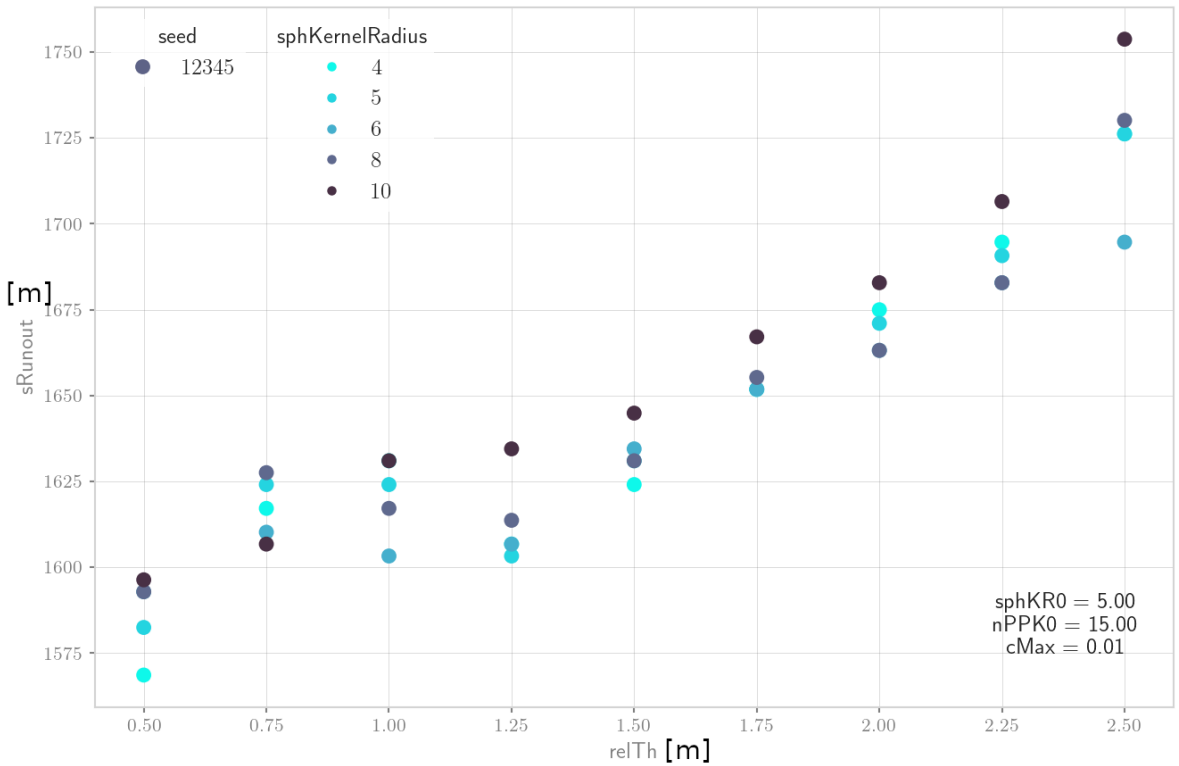
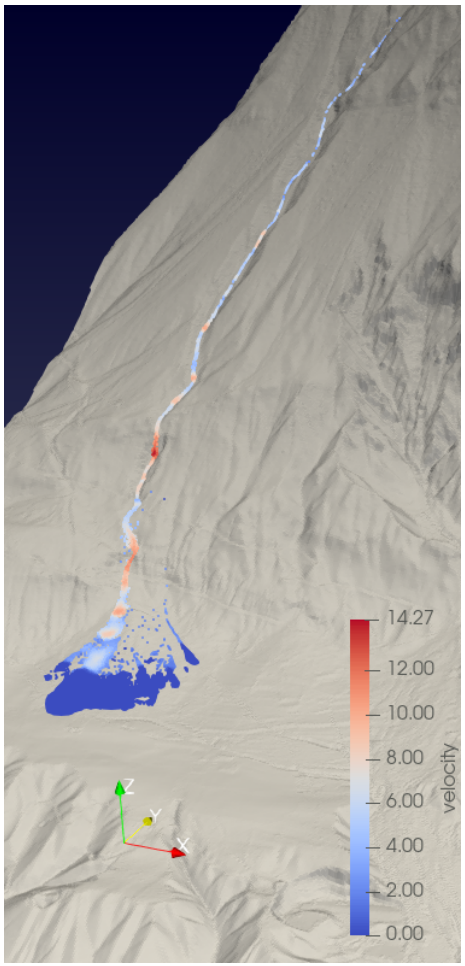
Assessing the influence of release thickness

We can also vary the inputs, for example release thickness.





Which shows us how the runout varies with an increasing release thickness.

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So how to check whether avalanche models are correct?

- Testing for convergence is a tool to check numerical correctness
- Semi analytical test cases give a reference when looking at flow variable evolution (thickness / velocity / ...)

These analyses allow for

- Quantification of the numerical uncertainty
- Necessary constraints / relations between input / discretization parameters

So numerical correctness can be checked, the remaining incorrectness needs to be quantified and communicated!