A comparison of gravitational effects between a spherical zonal band and a spherical shell discretized using tesseroids

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1. Introduction

Gravitational effects (e.g. gravitational potential (**GP**), gravity vector (**GV**), gravity gradient tensor (**GGT**), and gravitational or gravity curvatures (**GC**))

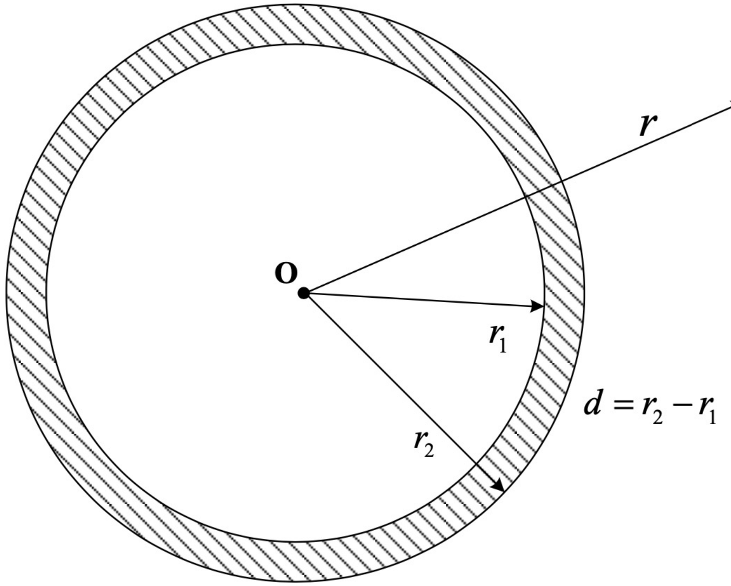

Fig. 1 Spherical shell

Revised from Deng and Shen (2018)

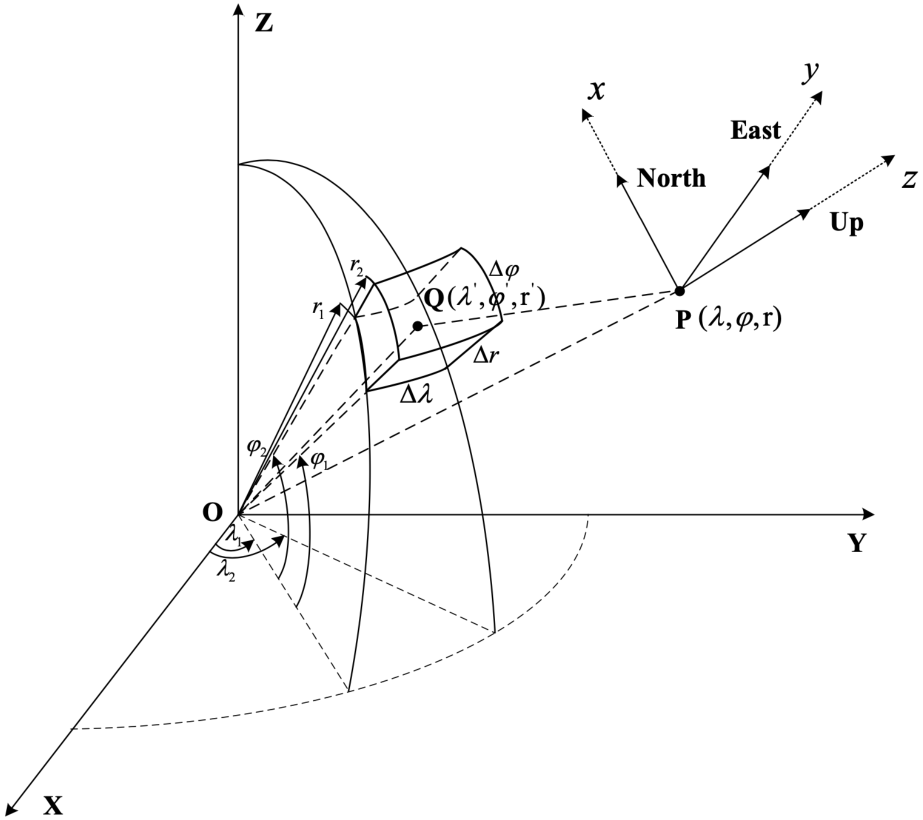

Fig. 2 Tesseroid

Revised from Deng and Shen (2009)

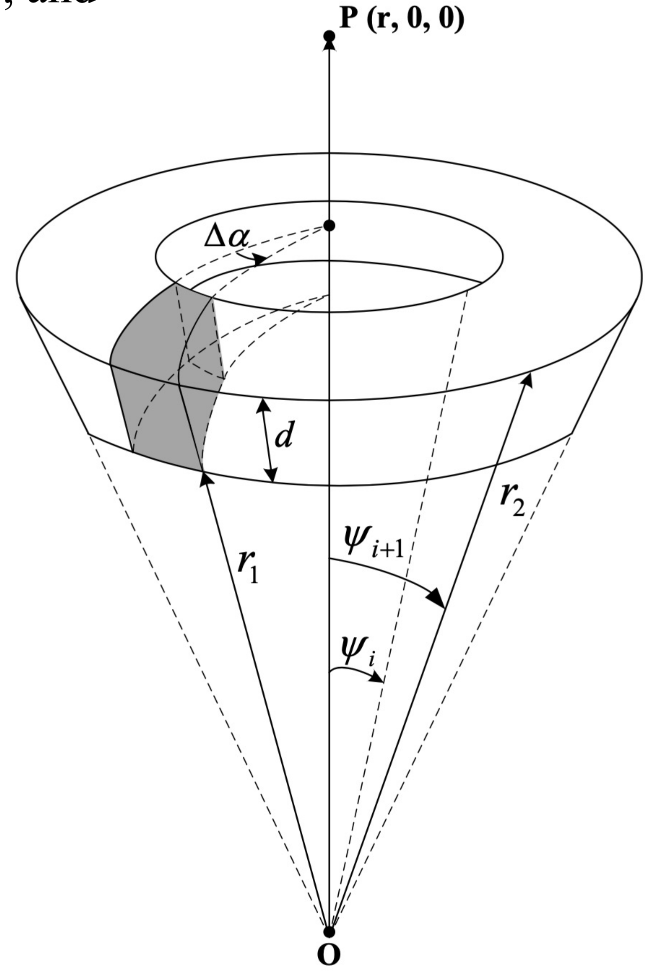

Fig. 3 Spherical zonal band

Revised from Heck and Seitz (2007)

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2. Theory: gravitational effects of a spherical cap

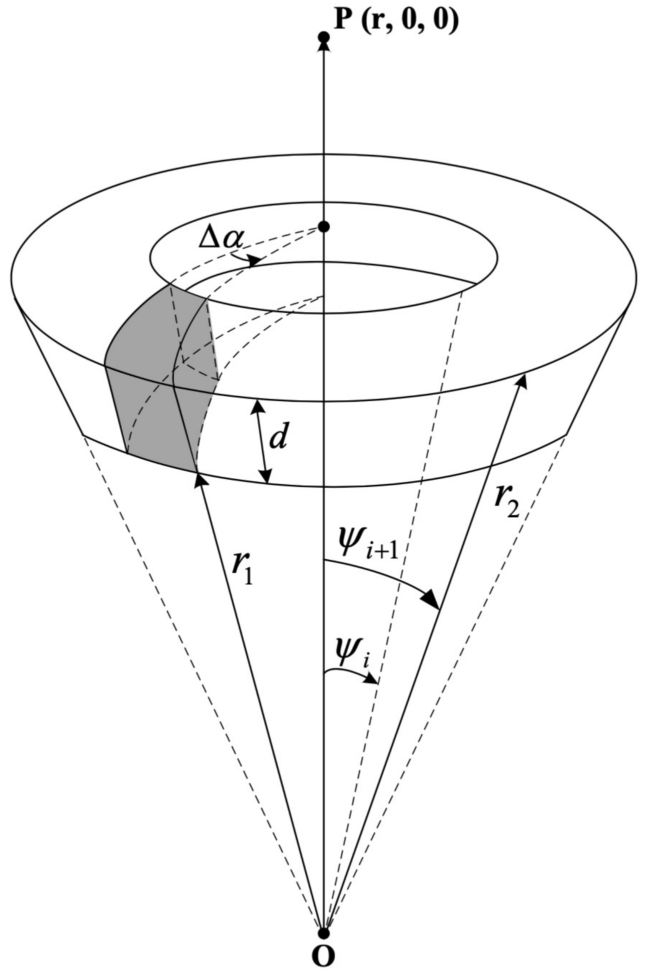

Fig. 3 Spherical zonal band

Revised from Heck and Seitz (2007)

The formula of the Gravitational Potential (GP) of a spherical cap is presented as (Papp and Wang 1996; Heck and Seitz 2007):

$$\begin{aligned} V(r;r_1,r_2,\psi_c) = 2\pi G\rho \Big[&\frac{1}{3r}\ell_c'^3 + \frac{1}{2}\ell_c' \cos\psi_c (r' - r\cos\psi_c) \\ &+ \frac{1}{2} r^2 \cos\psi_c \sin^2\psi_c \ln(\ell_c' + r' - r\cos\psi_c) \Big] \Big|_{r'=r_1}^{r'=r_2} \\ &+ 2\pi G\rho \left(+\frac{1}{3r} r'^3 - \frac{1}{2} r'^2 \right) \Big|_{r'=r_1}^{r'=r_2} \begin{cases} +1 & r \geqslant r_2 \\ -1 & r \leqslant r_1 \end{cases} \end{aligned} \tag{1}$$

$$\ell_c' = \sqrt{r^2 + r'^2 - 2rr'\cos\psi_c} \tag{2}$$

2. Theory

Gravity Vector (GV: Vx, Vy, Vz) and Gravity Gradient Tensor (GGT: Vxx, Vxy, Vxz, Vyy, Vyz, Vzz) of a spherical cap

$$V_x(r;r_1,r_2,\psi_c) = V_y(r;r_1,r_2,\psi_c) = 0 \quad (3)$$

$$\begin{aligned} V_z(r;r_1,r_2,\psi_c) &= \frac{\partial V(r;r_1,r_2,\psi_c)}{\partial r} \\ &= 2\pi G\rho\left\{\frac{1}{6r^2}\left[\ell_c'(r^2-3r^2\cos 2\psi_c-2r'^2)\right.\right. \\ &+ r\cos\psi_c\left[3r^2\sin^2\psi_c\left[2\ln(\ell_c'+r'-r\cos\psi_c)+1\right]\right. \\ &\left.\left.\left.-2\ell_c' r'\right]\right]\right\}\Bigg|_{r'=r_1}^{r'=r_2} + 2\pi G\rho\left(-\frac{r'^3}{3r^2}\right)\Bigg|_{r'=r_1}^{r'=r_2}\begin{cases}+1 & r\geqslant r_2\\ -1 & r\leqslant r_1\end{cases} \end{aligned} \quad (4)$$

v.s. Eq. (54) of Heck and Seitz (2007)

$$V_{xy}(r;r_1,r_2,\psi_c) = V_{xz}(r;r_1,r_2,\psi_c) = V_{yz}(r;r_1,r_2,\psi_c) = 0 \quad (5)$$

$$\begin{aligned} V_{zz}(r;r_1,r_2,\psi_c) &= \frac{\partial V^2(r;r_1,r_2,\psi_c)}{\partial r^2} = \frac{\partial V_z(r;r_1,r_2,\psi_c)}{\partial r} \\ &= 2\pi G\rho\left\{\frac{1}{6r^3\ell_c'}\left[r^4+2r^2r'^2+4r'^4-2r^2r'^2\cos^2\psi_c\right.\right. \\ &- 3r^4\cos 2\psi_c - r\cos\psi_c\left[r^2r'+4r'^3-3r^2r'\cos 2\psi_c\right. \\ &\left.\left.\left.-3r^2\sin^2\psi_c\left[2\ell_c'\ln(\ell_c'+r'-r\cos\psi_c)+3\ell_c'-2r'\right]\right]\right]\right\}\Bigg|_{r'=r_1}^{r'=r_2} \\ &+ 2\pi G\rho\left(\frac{2r'^3}{3r^3}\right)\Bigg|_{r'=r_1}^{r'=r_2}\begin{cases}+1 & r>r_2\\ -1 & r<r_1\end{cases} \end{aligned} \quad (6)$$

v.s. Eq. (19) of Lin et al. (2020)

$$V_{xx}(r;r_1,r_2,\psi_c) = V_{yy}(r;r_1,r_2,\psi_c) = -\frac{1}{2}V_{zz}(r;r_1,r_2,\psi_c) \quad (7)$$

2. Theory

Gravitational Curvatures
(GC: Vxxx, Vxxy, Vxxz, Vxyz,
Vyyx, Vyyy, Vyyz,
Vzzx, Vzzy, Vzzz)
of a spherical cap

$$
\begin{aligned}
&V_{xxx}(r;r_1,r_2,\psi_c) = V_{xxy}(r;r_1,r_2,\psi_c) = V_{xyz}(r;r_1,r_2,\psi_c) \\
&= V_{yyx}(r;r_1,r_2,\psi_c) = V_{yyy}(r;r_1,r_2,\psi_c) = V_{zzx}(r;r_1,r_2,\psi_c) \\
&= V_{zzy}(r;r_1,r_2,\psi_c) = 0
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
&V_{zzz}(r;r_1,r_2,\psi_c) = \frac{\partial V^3(r;r_1,r_2,\psi_c)}{\partial r^3} = \frac{\partial V_{zz}(r;r_1,r_2,\psi_c)}{\partial r} \\
&= 2\pi G\rho \Bigg\{ -\frac{1}{8r^4\ell_c'^3(\ell_c'+r'-r\cos\psi_c)} \Big[4r'(r^6+3r^4r'^2 \\
&+15r^2r'^4+4r'^6) + \ell_c'(r^6+3r^4r'^2+36r^2r'^4+16r'^6) \\
&-2r\cos\psi_c \Big[2r^4r'(\ell_c'+2r')+3r^2r'^3(3\ell_c'+10r') \\
&+8r'^5(3\ell_c'+4r')+r^6 \Big] + r^2 \Big[4r'^3\cos 2\psi_c(3r'\ell_c'+9r'^2+2r^2) \\
&+r\cos 3\psi_c(3r^2r'\ell_c'+6r^2r'^2+2r'^3\ell_c'-4r'^4+2r^4) \\
&-r^2\cos 4\psi_c(r^2\ell_c'+3r'^2\ell_c'+4r^2r'+4r'^3) \\
&+r^3r'\cos 5\psi_c(\ell_c'+2r') \Big] \Big] \Bigg\} \Bigg|_{r'=r_1}^{r'=r_2} \\
&+2\pi G\rho \left(-\frac{2r'^3}{r^4}\right) \Bigg|_{r'=r_1}^{r'=r_2} \begin{cases} +1 & r>r_2 \\ -1 & r<r_1 \end{cases}
\end{aligned} \tag{9}
$$

$$
V_{xxz}(r;r_1,r_2,\psi_c) = V_{yyz}(r;r_1,r_2,\psi_c) = -\frac{1}{2}V_{zzz}(r;r_1,r_2,\psi_c) \tag{10}
$$

2. Theory: gravitational effects of a spherical zonal band

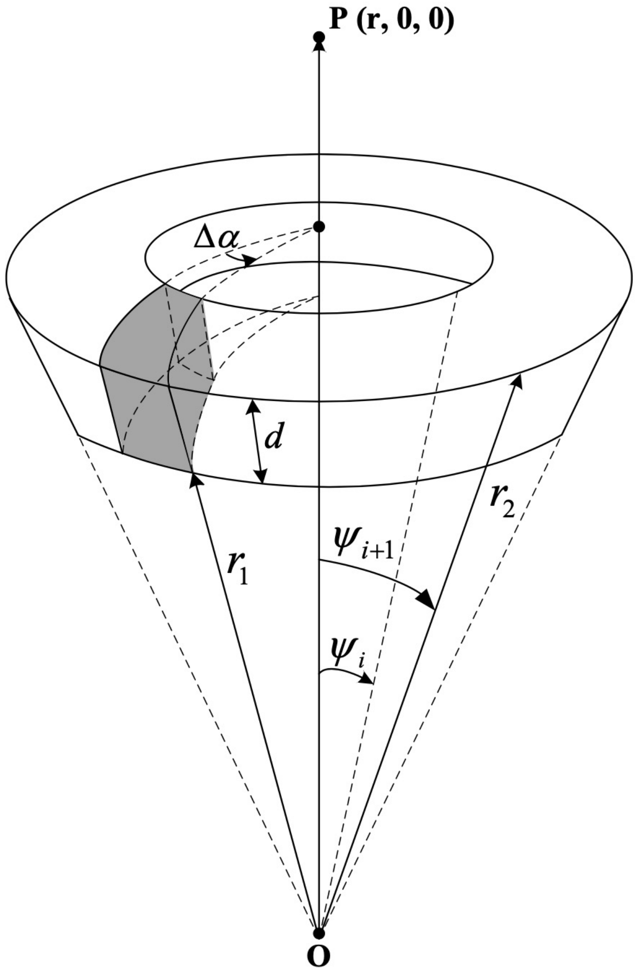

Fig. 3 Spherical zonal band

Revised from Heck and Seitz (2007)

$$F(r;r_1,r_2,\psi_i,\psi_{i+1}) = F(r;r_1,r_2,\psi_{i+1}) - F(r;r_1,r_2,\psi_i) \quad (11)$$

$$V_{xx}(r;r_1,r_2,\psi_i,\psi_{i+1}) = V_{yy}(r;r_1,r_2,\psi_i,\psi_{i+1}) = -\frac{1}{2}\left[V_{zz}(r;r_1,r_2,\psi_{i+1}) - V_{zz}(r;r_1,r_2,\psi_i)\right] \quad (12)$$

$$V_{xxz}(r;r_1,r_2,\psi_i,\psi_{i+1}) = V_{yyz}(r;r_1,r_2,\psi_i,\psi_{i+1}) = -\frac{1}{2}\left[V_{zzz}(r;r_1,r_2,\psi_{i+1}) - V_{zzz}(r;r_1,r_2,\psi_i)\right] \quad (13)$$

3. Numerical experiments

- The relative errors using **tesseroids** to discrete the whole **spherical zonal band** and **spherical shell** are performed by 3D Gauss-Legendre quadrature method (Wild-Pfeiffer 2008; Lin et al. 2020):

$$\delta F = Log_{10}\left(\left|\frac{\sum F^{\text{tess}}}{F^{\text{ref}}} - 1\right|\right) \tag{14}$$

Table 1 Numerical values in the experiments

Parameter	Notation	Magnitude	Unit
Newtonian gravitational constant	G	6.673×10^{-11}	$\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Up geocentric distance of the spherical cap	r_2	$6,378,137$	m
Down geocentric distance of the spherical cap	r_1	$6,377,137$	m
Thickness of the spherical cap	$d = r_2 - r_1$	1,000	m
Height above the surface of the spherical cap	h	260,000	m
Geocentric distance of the computation point P	$r = r_2 + h$	$6,638,137$	m
Density of the spherical cap	ρ	$2,670$	$\mathrm{kg\,m^{-3}}$

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➢ Comparison of computation time between a spherical zonal band and spherical shell discretized using tesseroids in double and quadruple precision with different grid sizes

Table 2 Numerical results of computation time

Grid size	N_band	N_shell	Ratio	GP_d	GV_d	GGT_d	GC_d	All_d	GP_q	GV_q	GGT_q	GC_q	All_q
$20^\circ \times 20^\circ$	18	162	9	6.3	7.5	5.7	5.9	6.3	8.7	8.7	8.8	8.8	8.8
$15^\circ \times 15^\circ$	24	288	12	7.2	9.4	8.3	8.5	8.4	11.7	11.7	11.7	11.7	11.7
$10^\circ \times 10^\circ$	36	648	18	15.9	13.8	13.7	13.8	14.3	17.4	17.9	17.7	17.8	17.7
$5^\circ \times 5^\circ$	72	2,592	36	31.3	33.5	31.7	31.3	31.9	35.7	35.7	35.6	35.5	35.6
$3^\circ \times 3^\circ$	120	7,200	60	53.4	55.3	56.2	55.5	55.1	60.3	59.7	59.2	59.6	59.7
$2^\circ \times 2^\circ$	180	16,200	90	85.7	86.0	83.0	84.6	84.8	89.7	88.7	91.3	89.4	89.8
$1^\circ \times 1^\circ$	360	64,800	180	173.5	174.9	173.9	173.5	174.0	178.7	179.5	179.5	180.1	179.4
$30' \times 30'$	720	259,200	360	333.5	347.8	352.4	354.9	347.2	359.8	360.4	361.5	362.7	361.1
$15' \times 15'$	1,440	1,036,800	720	730.3	719.2	707.0	709.0	716.4	716.4	721.5	725.3	724.9	722.0
$10' \times 10'$	2,160	2,332,800	1,080	1,084.3	1,094.4	1,079.0	1,087.2	1,086.2	1,069.9	1,082.5	1,092.0	1,090.1	1,083.6
$5' \times 5'$	4,320	9,331,200	2,160	2,149.9	2,165.3	2,177.0	2,168.9	2,165.3	2,143.0	2,133.9	2,156.2	2,178.0	2,152.8
$3' \times 3'$	7,200	25,920,000	3,600	3,571.1	3,690.6	3,653.0	3,642.0	3,639.4	3,597.9	3,609.2	3,619.9	3,647.9	3,618.7
$2' \times 2'$	10,800	58,320,000	5,400	5,421.8	5,487.6	5,447.4	5,439.2	5,449.0	5,338.4	5,360.5	5,448.0	5,416.2	5,390.8

$n^\circ \times n^\circ$ $180/n$

- The computation time of a spherical zonal band discretized using tesseroids is about **180/n** times less than that of a spherical shell discretized using tesseroids for gravitational effects (i.e., GP, GV, GGT, and GC) with different grid sizes in double and quadruple precision.

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- Comparison of computation errors between the tesseroids discretizing a **spherical zonal band** and the tesseroids discretizing a **spherical shell**

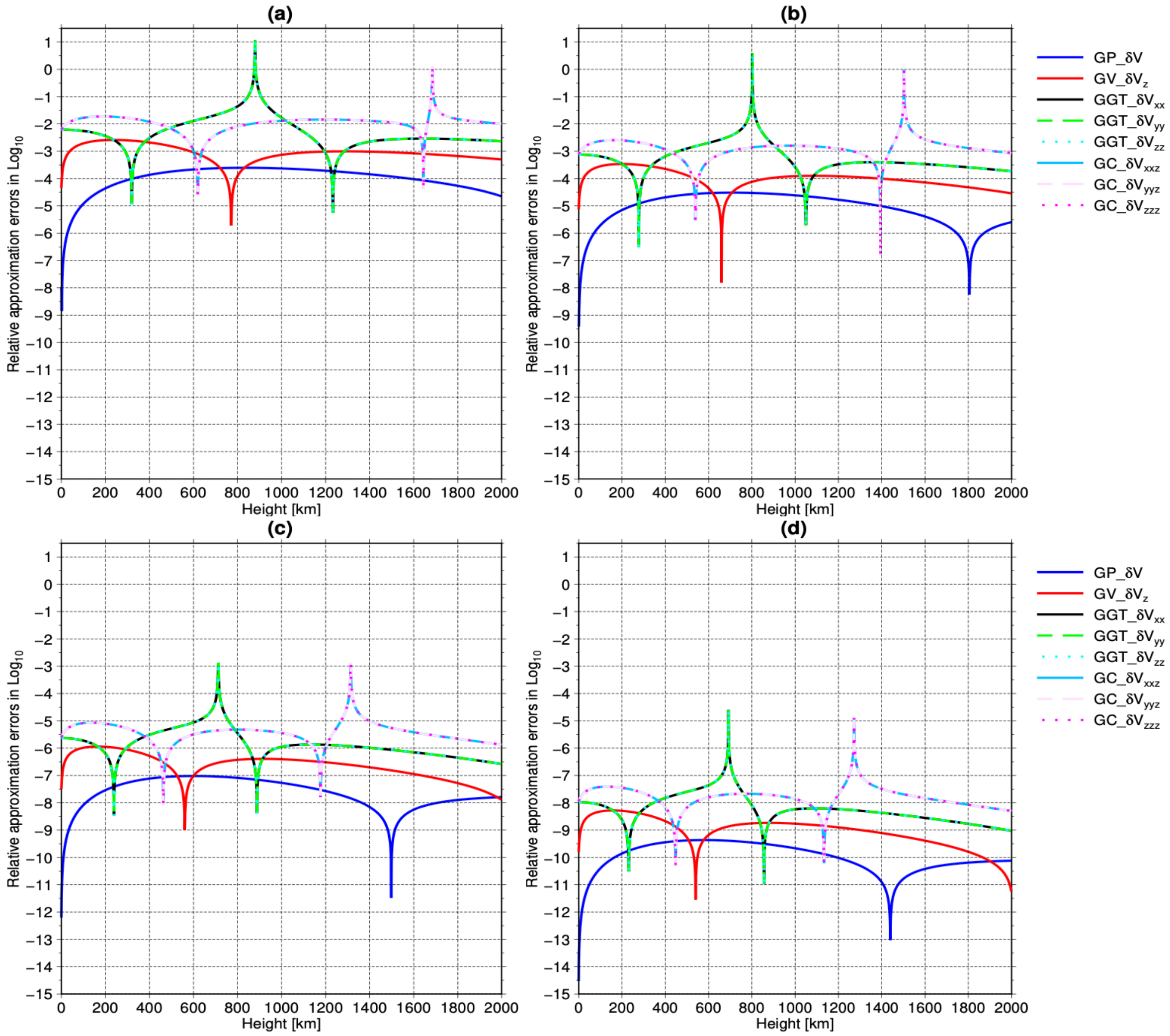

Fig. 4 Relative approximation errors in Log10 scale of nonzero gravitational effects of a **spherical zonal band** discretized using tesseroids with the influence of the computation point's height by a grid size of (a) 10d x 10d, (b) 5d x 5d, (c) 1d x 1d, and (d) 15'x15' in quadruple precision.

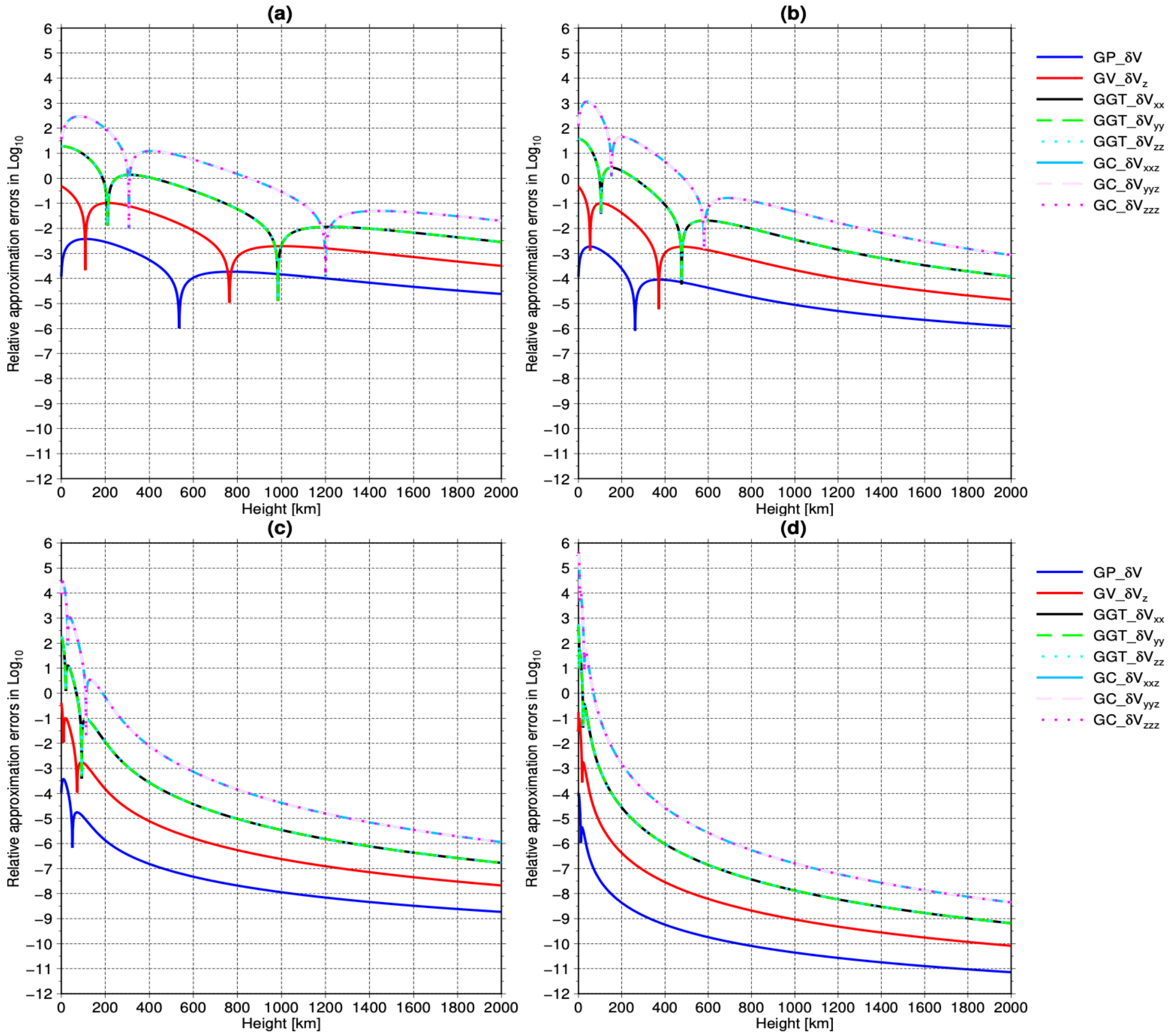

Fig. 5 Relative approximation errors in Log10 scale of nonzero gravitational effects of a **spherical shell** discretized using tesseroids with the influence of the computation point's height by a grid size of (a) 10d x 10d, (b) 5d x 5d, (c) 1d x 1d, and (d) 15'x15' in quadruple precision.

3. Numerical experiments

Table 3 Statistical information of the values in Fig. 4 for a **spherical zonal band**

Quantity	Grid size	Min	Max	Mean	STD
GP (δV)	$10^\circ \times 10^\circ$	-8.8	-3.6	-4.0	0.5
GP (δV)	$5^\circ \times 5^\circ$	-9.4	-4.5	-5.1	0.7
GP (δV)	$1^\circ \times 1^\circ$	-12.2	-7.0	-7.7	0.6
GP (δV)	$15' \times 15'$	-14.5	-9.4	-10.0	0.6
GV (δV_z)	$10^\circ \times 10^\circ$	-5.7	-2.6	-3.1	0.3
GV (δV_z)	$5^\circ \times 5^\circ$	-7.8	-3.5	-4.0	0.4
GV (δV_z)	$1^\circ \times 1^\circ$	-9.0	-5.9	-6.7	0.5
GV (δV_z)	$15' \times 15'$	-11.5	-8.3	-9.1	0.6
GGT ($\delta V_{xx}, \delta V_{yy}, \delta V_{zz}$)	$10^\circ \times 10^\circ$	-5.3	1.1	-2.3	0.6
GGT ($\delta V_{xx}, \delta V_{yy}, \delta V_{zz}$)	$5^\circ \times 5^\circ$	-6.5	0.6	-3.3	0.6
GGT ($\delta V_{xx}, \delta V_{yy}, \delta V_{zz}$)	$1^\circ \times 1^\circ$	-8.5	-2.9	-5.9	0.5
GGT ($\delta V_{xx}, \delta V_{yy}, \delta V_{zz}$)	$15' \times 15'$	-11.0	-4.6	-8.2	0.6
GC ($\delta V_{xxz}, \delta V_{yyz}, \delta V_{zzz}$)	$10^\circ \times 10^\circ$	-4.6	0.1	-2.0	0.3
GC ($\delta V_{xxz}, \delta V_{yyz}, \delta V_{zzz}$)	$5^\circ \times 5^\circ$	-6.8	-0.0	-2.9	0.4
GC ($\delta V_{xxz}, \delta V_{yyz}, \delta V_{zzz}$)	$1^\circ \times 1^\circ$	-8.0	-2.9	-5.4	0.4
GC ($\delta V_{xxz}, \delta V_{yyz}, \delta V_{zzz}$)	$15' \times 15'$	-10.3	-4.9	-7.8	0.4

Table 4 Statistical information of the values in Fig. 5 for a **spherical shell**

Quantity	Grid size	Min	Max	Mean	STD
GP (δV)	$10^\circ \times 10^\circ$	-6.0	-2.4	-3.8	0.6
GP (δV)	$5^\circ \times 5^\circ$	-6.1	-2.7	-4.9	0.9
GP (δV)	$1^\circ \times 1^\circ$	-8.7	-3.4	-7.6	1.1
GP (δV)	$15' \times 15'$	-11.1	-4.0	-10.0	1.2
GV (δV_z)	$10^\circ \times 10^\circ$	-5.0	-0.3	-2.5	0.9
GV (δV_z)	$5^\circ \times 5^\circ$	-5.2	-0.3	-3.4	1.1
GV (δV_z)	$1^\circ \times 1^\circ$	-7.7	-0.4	-6.1	1.5
GV (δV_z)	$15' \times 15'$	-10.1	-0.7	-8.5	1.6
GGT ($\delta V_{xx}, \delta V_{yy}, \delta V_{zz}$)	$10^\circ \times 10^\circ$	-4.9	1.3	-1.3	1.1
GGT ($\delta V_{xx}, \delta V_{yy}, \delta V_{zz}$)	$5^\circ \times 5^\circ$	-4.2	1.6	-2.2	1.4
GGT ($\delta V_{xx}, \delta V_{yy}, \delta V_{zz}$)	$1^\circ \times 1^\circ$	-6.8	2.3	-4.8	1.9
GGT ($\delta V_{xx}, \delta V_{yy}, \delta V_{zz}$)	$15' \times 15'$	-9.2	2.8	-7.2	2.0
GC ($\delta V_{xxz}, \delta V_{yyz}, \delta V_{zzz}$)	$10^\circ \times 10^\circ$	-3.9	2.5	-0.2	1.3
GC ($\delta V_{xxz}, \delta V_{yyz}, \delta V_{zzz}$)	$5^\circ \times 5^\circ$	-3.1	3.1	-1.0	1.7
GC ($\delta V_{xxz}, \delta V_{yyz}, \delta V_{zzz}$)	$1^\circ \times 1^\circ$	-5.9	4.5	-3.6	2.2
GC ($\delta V_{xxz}, \delta V_{yyz}, \delta V_{zzz}$)	$15' \times 15'$	-8.4	5.6	-6.0	2.4

- Mean values of the relative approximation errors in Log10 scale of the GP, GV, GGT, and GC for a **spherical zonal band** discretized using tesseroids with different grid sizes are **smaller** than those for a **spherical shell** discretized using tesseroids.

4. Conclusion

- The **simpler analytical expressions** for the radial GV and radial-radial GGT of a homogeneous spherical cap and spherical zonal band are derived. Moreover, we derive the **new analytical formulae of the GC** of a homogeneous spherical cap and spherical zonal band.
- Regarding the computation time and errors, **the benefit** of a spherical zonal band discretized using tesseroids is numerically confirmed in comparison with a spherical shell discretized using tesseroids.
- **A spherical zonal band discretized using tesseroids** can replace a spherical shell discretized using tesseroids in gravity field modelling in the future study.

5. Outlook

- The analytical solutions of gravitational effects of the spherical cap and spherical zonal band can be served as the reference values not only for tesseroids but also for **other spherical mass elements**.
- Research on the **higher-order gravitational potential gradients** of a spherical cap and spherical zonal band will be investigated based on the general formulae of Laplace’s equation for higher-order gravitational potential gradient.
- The homogeneous density for the gravitational effects of a spherical cap and spherical zonal band will be extended to the **variable density**.
- Future research on the **direct comparison** between the spherical zonal band and spherical shell in the context of a practical application like residual terrain correction will be performed.
- Moreover, the theoretical expressions of **magnetic curvatures** and **higher-order magnetic potential gradients** of a spherical cap and spherical zonal band will be investigated in the near future.

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