The System of Supercompact Equations for Two-Dimensional Waves Propagating on the Surface of a Three-Dimensional Deep Fluid

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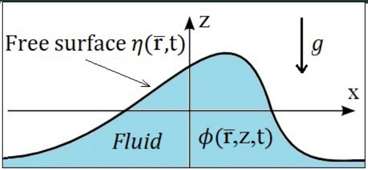
3D hydrodynamics with free surface

A 3D hydrodynamics of potential flow of an ideal incompressible deep fluid with a free surface in a presence of gravity field is considered:



$$\boxed{\triangle\phi = 0}$$

$$-\infty < x < \infty,\ -\infty < y < \infty,\ -\infty < z < \eta(x,y,t)$$

$$\vec{r} = (x, y)$$

$\eta(\vec{r}, t)$ — shape of the surface

$\psi(\vec{r}, t)$ — velocity potential at the surface

$\phi(\vec{r}, z, t)$ — velocity potential inside the fluid

Boundary conditions:

$$\left[\begin{array}{c} \dfrac{\partial\psi}{\partial t} + \dfrac{1}{2}|\nabla\psi|^2 + g\eta = 0 \\ \dfrac{\partial\eta}{\partial t} + \nabla\eta\nabla\psi = \dfrac{\partial\psi}{\partial z} \end{array}\right|_{\text{at } z = \eta(x,y,t)}$$

The system is Hamiltonian, η and ψ — Hamiltonian variables [V. E. Zakharov, 1968].

Hamiltonian system

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi},\ \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}$$

$$H = \frac{1}{2}\int dxdy \int_{-\infty}^{\eta} (\nabla\phi)^2\, dz + \frac{g}{2}\int \eta^2\, dxdy$$

Zakharov, V. E. (1968). Stability of periodic waves of finite amplitude on the surface of a deep fluid. Journal of Applied Mechanics and Technical Physics, 9(2), 190-194.

The Zakharov Equation

In the assumption of small wave steepness $\mu \ll 1$ the Hamiltonian can be expanded into the power series of η and ψ. Applying canonical transformation $\eta, \psi \to b, b^*$ one can simplify the original Hamiltonian by removing all so-called non-resonant terms:

Simplified Hamiltonian

$$H(b, b^*) = \int w_{\vec{k}} b_{\vec{k}} b^*_{\vec{k}} d\vec{k} + \frac{1}{2} \int T^{\vec{k}_2, \vec{k}_3}_{\vec{k}, \vec{k}_1} b^*_{\vec{k}} b^*_{\vec{k}_1} b_{\vec{k}_2} b_{\vec{k}_3} \delta_{\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3} d\vec{k} d\vec{k}_1 d\vec{k}_2 d\vec{k}_3$$

The equation of motion in this case is the traditional Zakharov equation:

$$\frac{\partial b_{\vec{k}}}{\partial t} + i \frac{\delta H}{\delta b_{\vec{k}}^*} = 0$$

$$i \frac{\partial b_{\vec{k}}}{\partial t} = w_{\vec{k}} b_{\vec{k}} + \int T^{\vec{k}_2, \vec{k}_3}_{\vec{k}, \vec{k}_1} b^*_{\vec{k}_1} b_{\vec{k}_2} b_{\vec{k}_3} \delta_{\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3} d\vec{k}_1 d\vec{k}_2 d\vec{k}_3$$

$T^{\vec{k}_2, \vec{k}_3}_{\vec{k}, \vec{k}_1}$ — four-wave interaction coefficient (very complicated in 2D).

The most compact form can be found in [V.V. Geogjaev, V.E. Zakharov, 2017].

Geogjaev V. V., Zakharov V. E. Numerical and analytical calculations of the parameters of power-law spectra for deep water gravity waves //JETP Letters. – 2017. – T. 106. – №. 3. – C. 184-187.

Four-wave interaction coefficient in 1D case

In 1D case coefficient $T_{k,k_1}^{k_2,k_3}$ has an explicit form [Dyachenko, 2020]:

$$T_{k,k_1}^{k_2,k_3} = \begin{cases} \frac{|kk_1k_2k_3|^{\frac{1}{4}}}{8\pi}\left[L_{-kk_1} + L_{-k_2k_3}\right] D_{k,k_1}^{k_2,k_3}, & \text{if all k > 0 (or < 0)} \\ \frac{|kk_1k_2k_3|^{\frac{1}{4}}}{8\pi}\left[-L_{kk_2} - L_{kk_3} - L_{k_1k_2} - L_{k_1k_3}\right] D_{k,k_1}^{k_2,k_3}, & \text{if} \quad kk_1 < 0 \quad \text{and} \quad k_2k_3 < 0 \\ 0, & \text{if} \quad kk_1k_2k_3 < 0 \end{cases}$$

Coefficients L_{kk_1} and $D_{k,k_1}^{k_2,k_3}$ are defined as follows:

$$L_{kk_1} = \frac{|kk_1| - kk_1}{\sqrt{|kk_1|}}$$

$$D_{k,k_1}^{k_2,k_3} = \begin{cases} \min(|k|, |k_1|, |k_2|, |k_3|), & kk_1k_2k_3 > 0 \\ 0, & kk_1k_2k_3 < 0 \end{cases}$$

Explicit form of $D_{k,k_1}^{k_2,k_3}$:

$$\begin{aligned} D_{k,k_1}^{k_2,k_3} &= \frac{1}{2}\left(|k| + |k_1| + |k_2| + |k_3|\right) - \frac{1}{4}\left(|k + k_1| + |k_2 + k_3|\right) - \\ &- \frac{1}{4}\left(|k - k_2| + |k - k_3| + |k_1 - k_2| + |k_1 - k_3|\right) \end{aligned}$$

Dyachenko A. I. Canonical system of equations for 1D water waves // Studies in Applied Mathematics. – 2020. – T. 144. – №. 4. – C. 493-503.

Resonant manifold

The coefficient $T_{k,k_1}^{k_2,k_3}$ has a special property allowing further simplification of the Hamiltonian by using canonical transformations:

$$T_{k,k_1}^{k_2,k_3} = 0 \quad \text{if} \quad kk_1k_2k_3 < 0, \tag{1}$$

Any canonical transformation applied to Hamiltonian system has to keep the value of coefficient $T_{k,k_1}^{k_2,k_3}$ unchanged on the resonant manifold:

$$\begin{aligned} k + k_1 &= k_2 + k_3, \\ \omega_k + \omega_{k_1} &= \omega_{k_2} + \omega_{k_3}, \end{aligned} \tag{2}$$

In 1D case all solutions of equations (2) can be divided into two parts: trivial and non-trivial solutions:

Trivial solutions:

$$\begin{aligned} k &= k_2, \\ k_1 &= k_3 \end{aligned}$$

(and permutations)

$$T_{k,k_1}^{k_2,k_3} = T_{k,k_1}^{k,k_1} = \frac{1}{2\pi} kk_1 \min(|k|, |k_1|)$$

Non-trivial solutions:

$$\begin{aligned} k &= a(1+\zeta)^2, \\ k_1 &= a(1+\zeta)^2\zeta^2, \\ k_2 &= -a\zeta^2, \\ k_3 &= a(1+\zeta+\zeta^2)^2, \\ &0 < \zeta < 1 \end{aligned}$$

$$kk_1k_2k_3 < 0 \Rightarrow T_{k,k_1}^{k_2,k_3} = 0$$

Canonical transformations: equation in b, b^*

One can apply canonical transformation to replace T^{k_2,k_3}_{k,k_1} by new four-wave interaction coefficient $\tilde{T}^{k_2,k_3}_{k,k_1}$ coinciding on the resonant manifold:

$$\tilde{T}^{k_2,k_3}_{k,k_1} = \begin{cases} \frac{1}{4\pi}\,(kk_1 + k_2k_3)\,D^{k_2,k_3}_{k,k_1}, & \text{if} \quad kk_1k_2k_3 > 0 \\ 0, & \text{if} \quad kk_1k_2k_3 < 0 \end{cases} \tag{3}$$

The Hamiltonian and the dynamical equation for b in k-space now are the following:

$$H(b, b^*) = \int w_k b_k b_k^* dk + \frac{1}{2}\int \tilde{T}^{k_2,k_3}_{k,k_1} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3$$

$$i\frac{\partial b_k}{\partial t} = w_k b_k + \int \tilde{T}^{k_2,k_3}_{k,k_1} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 .$$

Using operators $\hat{k}$ and $\hat{\omega}_k$ which are multiplication by $|k|$ and $\sqrt{g|k|}$ in Fourier space, the equation of motion in x-space takes the form:

$$\begin{aligned} i\frac{\partial b}{\partial t} \quad = \quad & \hat{\omega}_k b - \frac{1}{4}\Big[2b_x b^* \hat{k}(b_x) + b_x b_x \hat{k}(b^*) + \hat{k}(b_x b_x b^*) - \frac{\partial}{\partial x}(bb\hat{k}(b_x^*)) - \frac{\partial}{\partial x}\hat{k}(bbb_x^*) - \\ - \quad & 2\frac{\partial}{\partial x}(b_x^* b\hat{k}(b))\Big] + \frac{1}{4}\Big[b^*\hat{k}(b_x b_x) - \frac{\partial}{\partial x}(b_x^*\hat{k}(bb))\Big] + \frac{1}{2}\Big[b_x\hat{k}(b_x b^*) - \frac{\partial}{\partial x}(b\hat{k}(b_x^* b))\Big] \end{aligned}$$

Canonical transformations: equations in c^+ and c^-

The diagonal part of four-wave interaction coefficient on the resonant manifold can be written as:

$$T^{k,k_1}_{k,k_1} = \begin{cases} \frac{1}{2\pi}|k||k_1|\min(|k|,|k_1|), & \text{if all } k \text{ are positive (or negative)} \\ -\frac{1}{2\pi}|k||k_1|\min(|k|,|k_1|), & \text{if } kk_1 < 0 \text{ and } k_2k_3 < 0 \\ 0 & \text{if} \quad kk_1k_2k_3 < 0 \end{cases} \tag{4}$$

Now we replace the original coefficient by the new coefficient $\tilde{\tilde{T}}^{k_2,k_3}_{k,k_1}$:

$$\tilde{\tilde{T}}^{k_2,k_3}_{k,k_1} = \frac{1}{2\pi}\sqrt{|k||k_1||k_2||k_3|}D^{k,k_1}_{k,k_1} \tag{5}$$

The canonical transformation ($b \to c$) replaces T^{k_2,k_3}_{k,k_1} by $\tilde{\tilde{T}}^{k_2,k_3}_{k,k_1}$ and allow to divide waves into two groups: waves running to the left and to the right:

$$c(x,t) = c^+(x,t) + c^-(x,t)$$

Dyachenko A. I. Canonical system of equations for 1D water waves //Studies in Applied Mathematics. – 2020. – T. 144. – №. 4. – C. 493-503.

Canonical transformations: equations in c^+ and c^-

Hamiltonian in terms of c^+ and c^- can be written in x-space:

$$\begin{aligned} H(c^+, c^-) = & \int c^{+*}\hat{V}c^+\,dx + \frac{1}{2}\int |c^+|^2\left[\frac{i}{2}(c^+ c_x^{+*} - c^{+*}c_x^+) - \hat{k}|c^+|^2\right]dx + \\ & + \int c^{-*}\hat{V}c^-\,dx + \frac{1}{2}\int |c^-|^2\left[\frac{i}{2}(c^{-*} c_x^{-} - c^{-}c_x^{-*}) - \hat{k}|c^-|^2\right]dx + \\ & + \int\left[|c^+|^2\hat{k}|c^-|^2 + c^{+*}c^{-*}\hat{k}(c^+c^-) + i(c^{+*}c^-)\frac{\partial}{\partial x}(c^+c^-)\right]dx \end{aligned}$$

Corresponding equations of motion are supercompact equations:

$$\frac{\partial c^+}{\partial t} + \partial_x^+\frac{\delta H}{\delta c^{+*}} = 0, \qquad \frac{\partial c^-}{\partial t} - \partial_x^-\frac{\delta H}{\delta c^{-*}} = 0$$

$$\frac{\partial c^+}{\partial t} + i\hat{\omega}c^+ = \partial_x^+\left[i(|c^+|^2 - |c^-|^2)c_x^+ + c^+\hat{k}(|c^+|^2 - |c^-|^2) - ic^+c^-c_x^{-*} - c^{-*}\hat{k}\left(c^+c^-\right)\right]$$
$$\frac{\partial c^-}{\partial t} + i\hat{\omega}c^- = \partial_x^-\left[i(|c^-|^2 - |c^+|^2)c_x^- - c^-\hat{k}(|c^-|^2 - |c^+|^2) - ic^-c^+c_x^{+*} + c^{+*}\hat{k}\left(c^+c^-\right)\right]$$

$\hat{V} = \hat{\omega}/\hat{k}$, operators $\hat{k}$ and $\hat{\omega}$ are multiplication by $|k|$ and $\sqrt{g|k|}$ in k-space, ∂_x^+ and ∂_x^- — by $ik\theta_k$ and $ik\theta_{-k}$ where θ_k — Heaviside step function.

To 2D case: equation in b

To generalize the coefficent to the case of 2D waves we replace wave numbers by vectors $k \to \vec{k}$, coefficient $D_{k,k_1}^{k_2,k_3} \to D_{\vec{k},\vec{k}_1}^{\vec{k}_2,\vec{k}_3}$ and the products of wave numbers by scalar products, and thus taking into account angular dependence:

$$\tilde{T}_{k,k_1}^{k_2,k_3} = \frac{1}{4\pi}(kk_1 + k_2k_3)\,D_{k,k_1}^{k_2,k_3} \quad \xrightarrow{2D} \quad \tilde{T}_{\vec{k},\vec{k}_1}^{\vec{k}_2,\vec{k}_3} = \frac{1}{4\pi}\left(\vec{k}\cdot\vec{k}_1 + \vec{k}_2\cdot\vec{k}_3\right) D_{\vec{k},\vec{k}_1}^{\vec{k}_2,\vec{k}_3}$$

$$D_{\vec{k}\,\vec{k}_1}^{\vec{k}_2\vec{k}_3} = \frac{1}{2}(|\vec{k}|+|\vec{k}_1|+|\vec{k}_2|+|\vec{k}_3|) - \frac{1}{4}(|\vec{k}+\vec{k}_1|+|\vec{k}_2+\vec{k}_3|) - \frac{1}{4}(|\vec{k}-\vec{k}_2|+|\vec{k}-\vec{k}_3|+|\vec{k}_1-\vec{k}_2|+|\vec{k}_1-\vec{k}_3|)$$

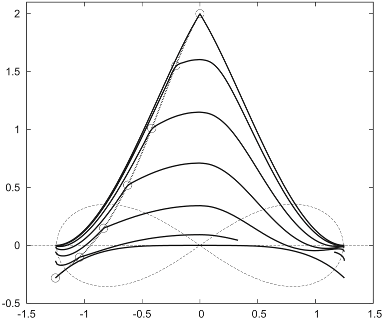
Now $b = b(x, y, t)$, operator $\hat{k}$ now is multiplication by $|\vec{k}| = \sqrt{k_x^2 + k_y^2}$, $\hat{\omega}$ — by $\sqrt{g|\vec{k}|}$ in k-space, and partial derivatives are replaced by ∇ operators.

1D Equation of motion in b and b^* variables turns to the:

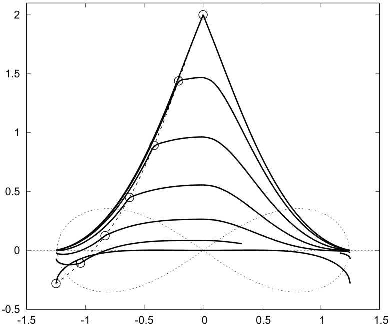
$$\begin{aligned}
i\frac{\partial b}{\partial t} \quad &= \quad \hat{\omega}_{\vec{k}} b - \frac{1}{4}\Big[2(\nabla b)\cdot\hat{k}(\nabla b)b^* + (\nabla b)\cdot(\nabla b)\hat{k}(b^*) + \hat{k}((\nabla b)\cdot(\nabla b)b^*) - \\
&- \quad \nabla\cdot(bb\hat{k}(\nabla b^*)) - \nabla\cdot(\hat{k}(bb(\nabla b^*))) - 2\nabla\cdot((\nabla b^*)b\hat{k}(b))\Big] + \\
&+ \quad \frac{1}{4}\Big[b^*\hat{k}((\nabla b)\cdot(\nabla b)) - \nabla\cdot((\nabla b^*)\hat{k}(bb))\Big] + \\
&+ \quad \frac{1}{2}\Big[(\nabla b)\cdot\hat{k}((\nabla b)b^*) - \nabla\cdot(b\hat{k}((\nabla b^*)b))\Big] \qquad (6)
\end{aligned}$$

The comparison of coefficients in 2D case

Comparison of the original two-dimensional Zakharov coefficient $T_{\vec{k},\vec{k}_1}^{\vec{k}_2,\vec{k}_3}$ with the generalized coefficient $\tilde{T}_{\vec{k},\vec{k}_1}^{\vec{k}_2,\vec{k}_3}$ on the resonance manifold was carried out by Dr. V.V. Geogjaev. The figures show the behaviour of coefficients on Phillips curve respectively.



$T_{\vec{k},\vec{k}_1}^{\vec{k}_2,\vec{k}_3}$



$\tilde{T}_{\vec{k},\vec{k}_1}^{\vec{k}_2,\vec{k}_3}$

The comparison showed that the coefficients are very similar which justifies the generalization.

To 2D case: equation in c^+ and c^-

The same procedure can be performed for equations in c^+ and c^-. Now we also replace coefficient $D^{k_2,k_3}_{k,k_1} \to D^{\vec{k}_2,\vec{k}_3}_{\vec{k},\vec{k}_1}$, $\partial^+_x \to i\hat{k}\theta_{k_x}$, $\partial^-_x \to -i\hat{k}\theta_{-k_x}$.

Now $c^+ = c^+(x,y,t)$, $c^- = c^-(x,y,t)$ and the Hamiltonian takes the following form in x-space:

$$H = \int c^{+*}\hat{V}c^+\,dxdy + \frac{1}{2}\int \left[\frac{1}{4}(c^{+2}\hat{k}c^{+*2} + c^{+*2}\hat{k}c^{+2}) - |c^+|^2\hat{k}|c^+|^2\right] dxdy +$$
$$+ \int c^{-*}\hat{V}c^-\,dxdy + \frac{1}{2}\int \left[\frac{1}{4}(c^{-*2}\hat{k}c^{-2} + c^{-2}\hat{k}c^{-*2}) - |c^-|^2\hat{k}|c^-|^2\right] dxdy +$$
$$+ \int \left[|c^+|^2\hat{k}|c^-|^2 + c^{+*}c^{-*}\hat{k}(c^+c^-) - i(c^{+*}c^-)\hat{k}(c^+c^{-*})\right] dxdy$$

The corresponding equations of motion now are:

$$i\frac{\partial c^+}{\partial t} = \hat{k}^+\frac{\delta H}{\delta c^{+*}}, \qquad i\frac{\partial c^-}{\partial t} = -\hat{k}^-\frac{\delta H}{\delta c^{-*}}, \quad \text{here } \hat{k}^+ = \hat{k}\theta_{k_x},\ \hat{k}^- = -\hat{k}\theta_{-k_x}$$

$$\frac{\partial c^+}{\partial t} = -i\hat{\omega}c^+ - i\hat{k}^+\left[\frac{1}{2}c^{+*}\hat{k}(c^{+2}) - c^+\hat{k}(|c^+|^2 - |c^-|^2) - c^-\hat{k}(c^+c^{-*}) + c^{-*}\hat{k}\,(c^+c^-)\right]$$
$$\frac{\partial c^-}{\partial t} = -i\hat{\omega}c^- + i\hat{k}^-\left[\frac{1}{2}c^{-*}\hat{k}(c^{-2}) - c^-\hat{k}(|c^-|^2 - |c^+|^2) - c^+\hat{k}(c^{+*}c^-) + c^{+*}\hat{k}\,(c^+c^-)\right] \quad (7)$$

Integrals of motion:

$$N^+ = \int \frac{|c^+_k|^2}{|k|}d\vec{k},\ N^- = \int \frac{|c^-_k|^2}{|k|}d\vec{k},\ P_x = \int \frac{k_x}{|k|}(|c^+|^2+|c^-|^2)d\vec{k},\ P_y = \int \frac{k_y}{|k|}(|c^+|^2+|c^-|^2)d\vec{k}$$

Numerical simulations

As a physical problem to test our models we consider 1D perturbed standing wave $\lambda_0 = 20m., \mu = |\vec{\nabla}\eta| \approx 0.26$ in a water channel with vertical smooth walls:

$$\frac{\partial \eta}{\partial x}|_{x=0,L} = 0; \quad \eta_k = \sum_k a_k cos(kx) \tag{8}$$

In terms of b and c^+, c^- the surface can be recovered by following canonical transformations:

$$\eta_k = \frac{|\vec{k}|^{\frac{1}{4}}}{\sqrt{2}g^{\frac{1}{4}}}\left[b_{\vec{k}} + b^*_{\vec{k}}\right]; \quad \eta_k = \frac{|\vec{k}|^{-\frac{1}{4}}}{\sqrt{2}g^{\frac{1}{4}}}\left[c^+_{\vec{k}} + c^-_{\vec{k}} + c^{+\,*}_{\vec{k}} + c^{-\,*}_{\vec{k}}\right] \tag{9}$$

This results in the following conditions for variables:

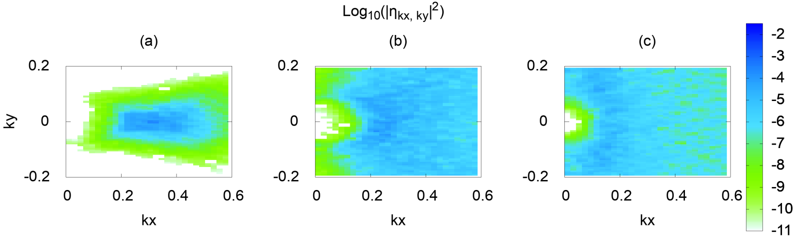
$$b(-k_x, k_y) = b(k_x, k_y); \quad c^-(-k_x, k_y) = c^+(k_x, k_y) \tag{10}$$

Thus, condition (10) for b variables is maintained at each time step while calculating in equation (6). In the case of variables c^+, c^- one can use only the first equation in (7) with condition (10) for c^+, c^- variables:

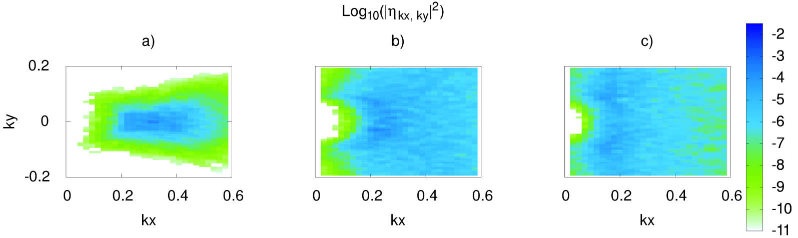
$$\frac{\partial c^+}{\partial t} = -i\hat{\omega}c^+ - i\hat{k}^+\left[\frac{1}{2}c^{+\,*}\hat{k}(c^{+\,2}) - c^+\hat{k}(|c^+|^2 - |c^-|^2) - c^-\hat{k}(c^+c^{-\,*}) + c^{-\,*}\hat{k}\left(c^+c^-\right)\right]$$

2D Spectra dynamics

Equation in b, b^*:

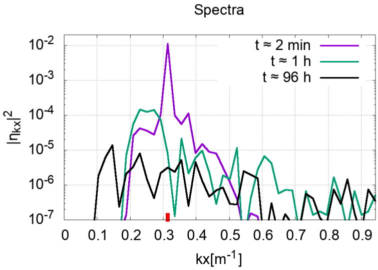


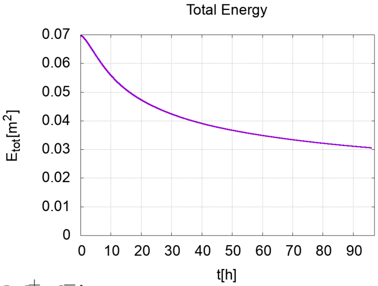
Equations in c^+, c^-:



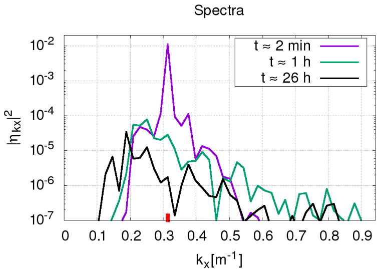
1D spectra and energy dynamics

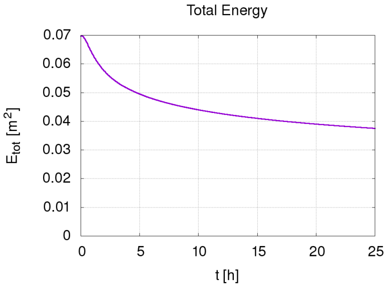
Equation in b, b^*:





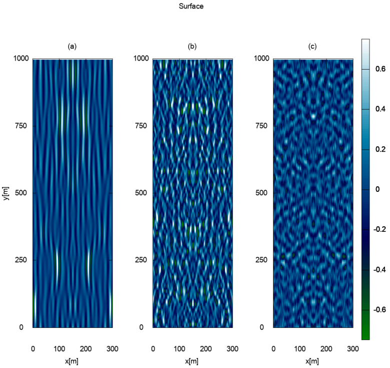
Equations in c^+, c^-:

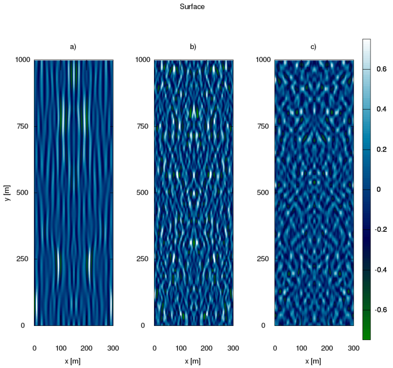




Surface dynamics

Equation in b, b^* Equations in c^+, c^-





Thus, the results obtained in both proposed models are very similar. Consequently, the 2D supercompact equations should also be very similar to the original 2D Zakharov equation.

Conclusion

- Two new Hamiltonian models have been proposed to describe the dynamics of 2D deep water waves. The derivation of the models was based on the use of compact forms of the 1D Zakharov equation.
- As a physical problem the dynamics of standing waves in a water channel with smooth vertical walls was considered. Similarities in overall dynamics in both models and comparison with original 2D Zakharov equation allow us to conclude that these models can be successfully applied to numerical simulations of 2D deep water waves.

Thank you for your attention!