A Semi-Lagrangian scheme for the free surface Euler system with application to rotational wave flows in general bathymetry

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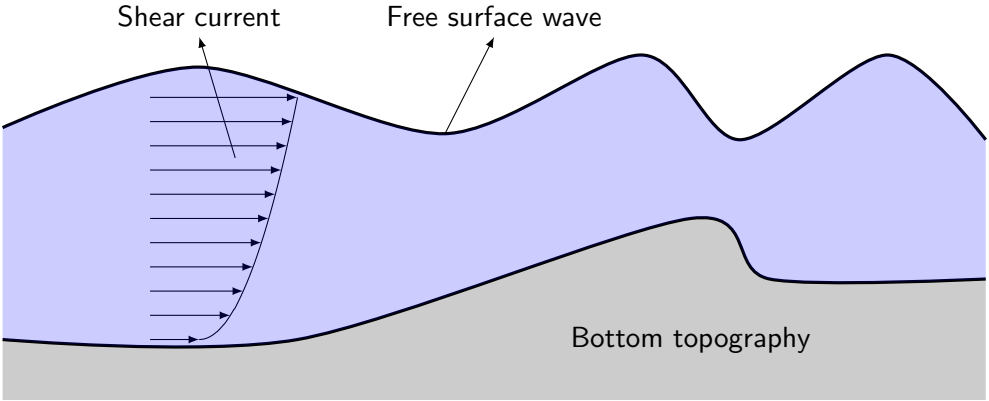
The physical problem



The mathematical modeling conundrum

Mathematical complexity ⟷ (vs) Computational Efficiency

Tailor made models ⟷ (vs) Universal methods

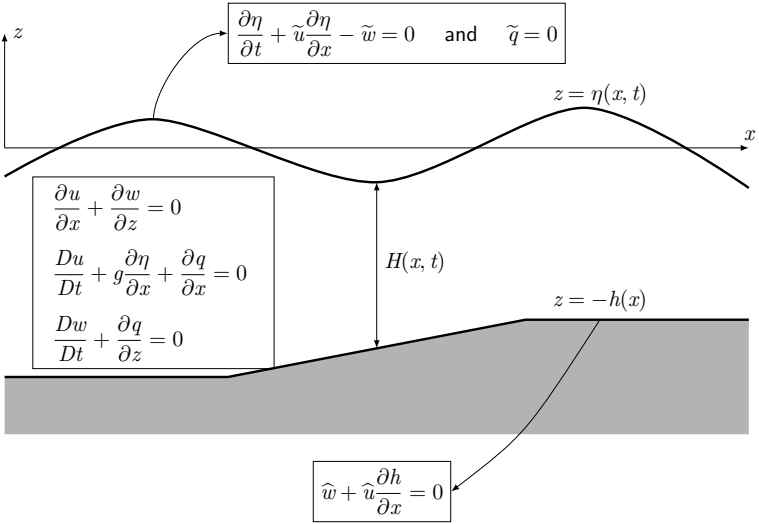
But what about robustness?

↓

Splitting Schemes → Block structure

Splitting Schemes → Incorporation of known techniques

The Free-Surface Euler system



Generalized layer coordinates

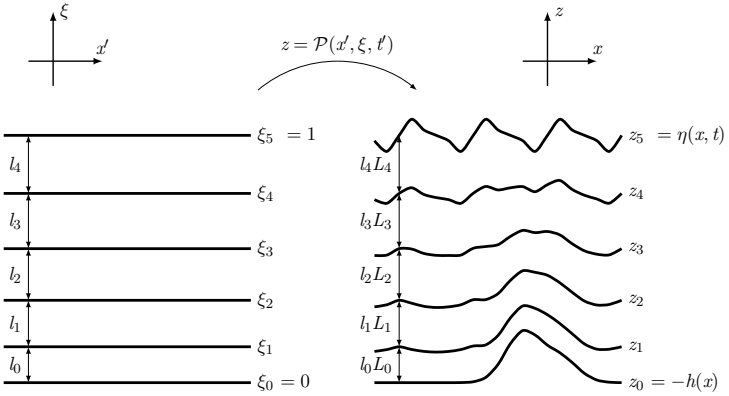
$$\left\{\begin{array}{c} z = \mathcal{P}(x', \xi, t') \\ x = x' \\ t = t' \end{array}\right\}$$

$$\Downarrow$$

$$Z_t \stackrel{\text{def}}{=} \frac{\partial \mathcal{P}}{\partial t}$$

$$Z_x \stackrel{\text{def}}{=} \frac{\partial \mathcal{P}}{\partial x}$$

$$L \stackrel{\text{def}}{=} \frac{\partial \mathcal{P}}{\partial \xi}$$



$\boxed{\vartheta = w - uZ_x - Z_t} \longrightarrow$ Vertical velocity through the moving grid

The Generalized Coordinate Euler system

Using $U = \begin{bmatrix} L & Lu & Lw \end{bmatrix}^T$ the GCE becomes:

The compact GCE

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x} F(U) + \frac{\partial}{\partial \xi} G(U;\vartheta) + Q(q,L) = S(U) \tag{2a}$$

$$div(U) = 0 \tag{2b}$$

$$\text{BC: } \widetilde{\vartheta} = \widehat{\vartheta} = \widetilde{q} = \widehat{w} - \widehat{u}\widehat{Z_x} = 0 \tag{2c}$$

$$F(U) = \begin{bmatrix} Lu & Lu^2 + \frac{1}{2} g L^2 & Luw \end{bmatrix}^T$$

$$S(U) = \begin{bmatrix} 0 & gL\frac{\partial}{\partial x}(L - \eta) & 0 \end{bmatrix}^T$$

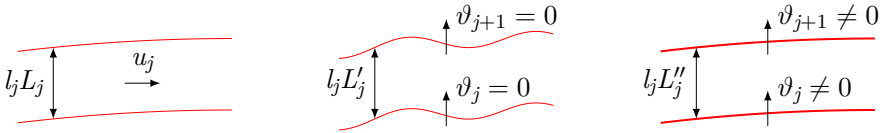
$$G(U;\vartheta) = \begin{bmatrix} \vartheta & \vartheta u & \vartheta w \end{bmatrix}^T$$

$$Q(q,L) = \begin{bmatrix} 0 & \frac{\partial}{\partial x}(Lq) - \frac{\partial}{\partial \xi}(Z_x q) & \frac{\partial q}{\partial \xi} \end{bmatrix}^T$$

$$div(U) = \frac{\partial}{\partial x}(Lu) + \frac{\partial}{\partial \xi}(w - uZ_x)$$

Layer kinematics

$$\frac{\partial L}{\partial t} + \frac{\partial}{\partial x}(Lu) = 0 \qquad \frac{\partial L}{\partial t} + \frac{\partial \vartheta}{\partial \xi} = 0$$



Initial Lagrangian Remeshed

$$\underbrace{\begin{array}{c}\text{Lagrangian step added to } x \text{ advection} \\ \text{Remeshing step added to } \xi \text{ advection}\end{array}}$$

$$\Downarrow$$

Semi-Lagrangian splitting scheme (SLS)

Operator Splitting

Step **(1/3)**: Multilayer Shallow Water Equations (mSWE) with constant dynamic pressure:

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x} F(U) + Q(q^n, L) = S(U) \qquad U^n \longrightarrow U^* \tag{5}$$

Step **(2/3)**: Vertical Remeshing Operator (VRO):

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial \xi} G(U; \vartheta) = 0 \qquad U^* \longrightarrow U^{**} \tag{6}$$

Step **(3/3)**: Pressure Correction Operator (PCO):

$$\left\{ \begin{array}{c} \dfrac{\partial U}{\partial t} + Q(\delta q^n, L^{n+1}) = 0 \\ div(U^{n+1}) = 0 \end{array} \right\} \qquad \left\{ \begin{array}{c} U^{**} \longrightarrow U^{n+1} \\ q^{n+1} = q^n + \delta q^n \end{array} \right\} \tag{7}$$

Numerical scheme

Step **(1/3)**: Multilayer Shallow Water Equations (mSWE) with constant dynamic pressure:

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x} F(U) + Q(q^n, L) = S(U) \qquad U^n \longrightarrow U^* \tag{8}$$

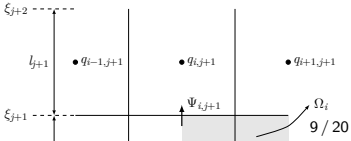
- Temporal $\rightarrow$ 4th order *Runge-Kutta*.
- Spatial $\rightarrow$ Second order *MUSCL*-type Finite Volume scheme.
- Flux $\rightarrow$ Approximate solver of [Roe, 1981]

Step **(2/3)**: Vertical Remeshing Operator (VRO):

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial \xi} G(U; \vartheta) = 0 \qquad U^* \longrightarrow U^{**} \tag{9}$$

- First order explicit Finite Volume scheme
- Flux $\rightarrow$ One speed scheme of [Rusanov, 1961]

Step **(3/3)**: Pressure Correction Operator (PCO):



Numerical wave tank

Wave generation $\rightarrow$ relaxation zone blending:

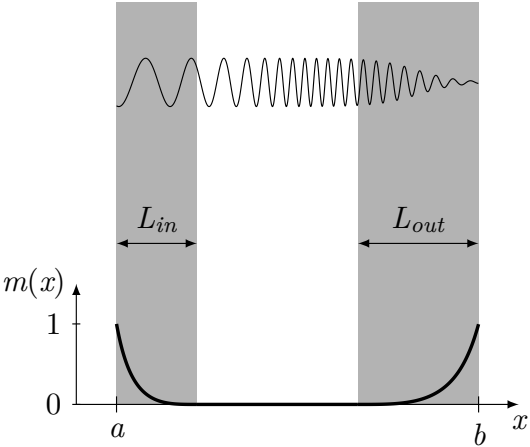
$$(1-m)\left[\frac{U-U_0}{\Delta t}+R\right]+m\left[\frac{U-U_{zone}}{\Delta t}\right]=0$$

$$\Longrightarrow U=(1-m)\,U^*+mU_{zone} \qquad (11)$$

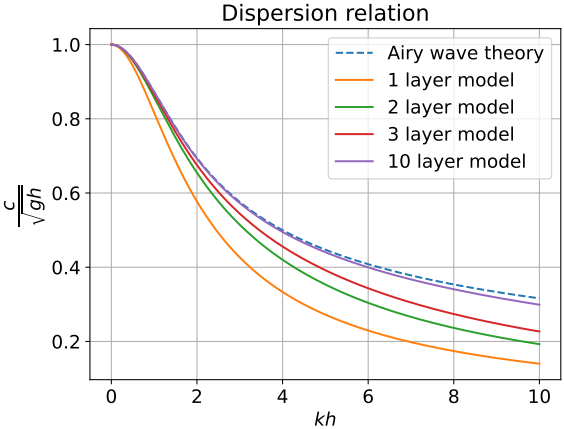
where $U^* = U_0 - \Delta t \cdot R$

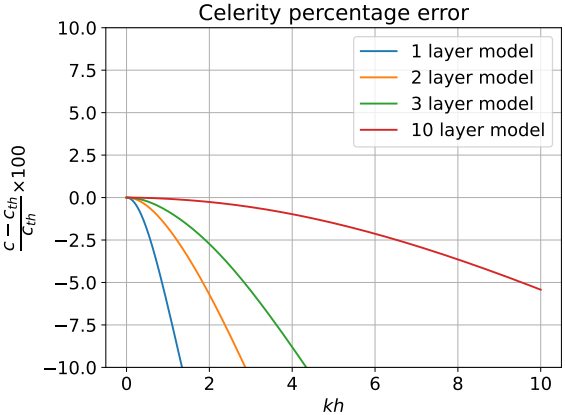
$U_{zone}(x,t) \rightarrow$ target solution

$m(x) \rightarrow$ blending function.



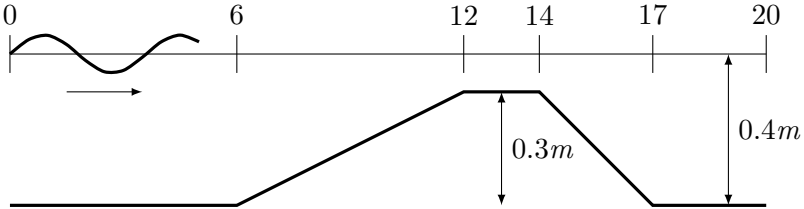
Linear dispersion properties





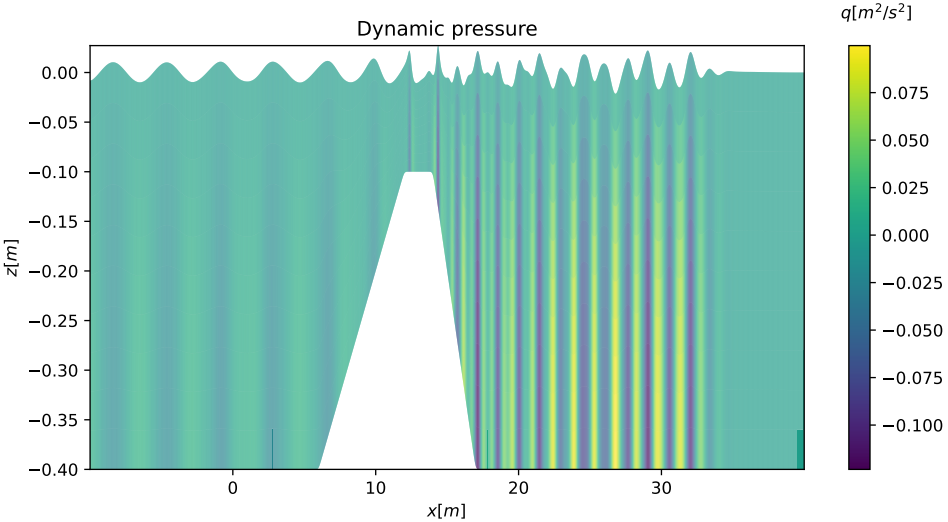
Comparison with Airy theory: $\dfrac{c_{Airy}^2}{gh} = \dfrac{\tanh kh}{kh}$

Propagation over submerged trapezoidal bar

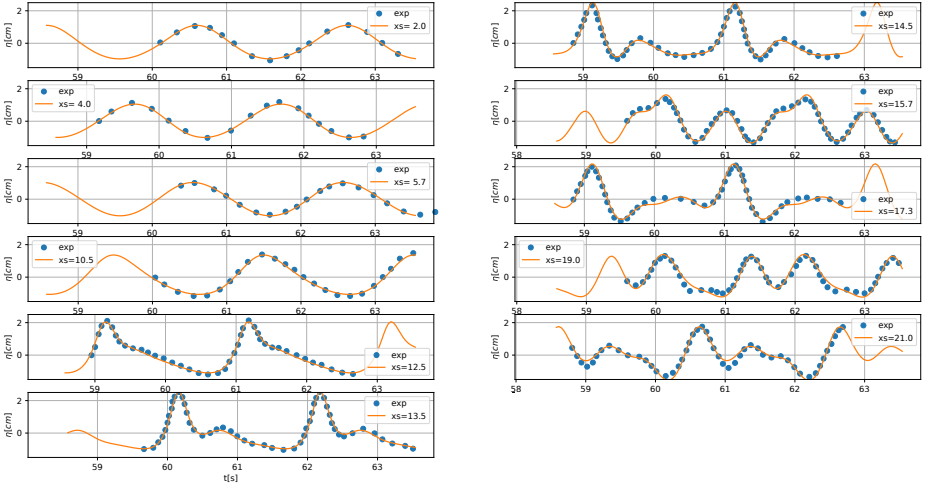


- Wave $\rightarrow$ Amplitude $A = 0.01m$ and length $\lambda = 3.7407m$
- Horizontal $\rightarrow$ $N_x = 3000$ cells of equal size.
- Vertical $\rightarrow$ $N_l = 10$ layers with $l_j = 1/N_l$.
- Timestepping $\rightarrow$ $\mathbb{CFL} = 0.9$
- Comparison with experimental data at stations [Dingemans, 1994].

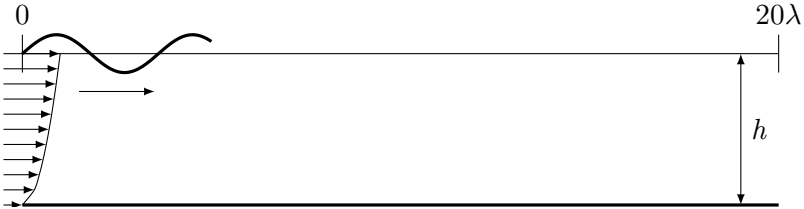
Propagation over submerged trapezoidal bar



Propagation over submerged trapezoidal bar

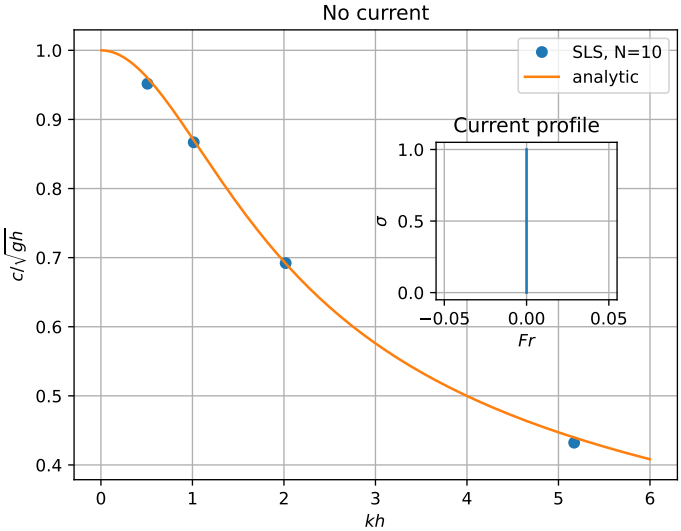


Wave-current interaction

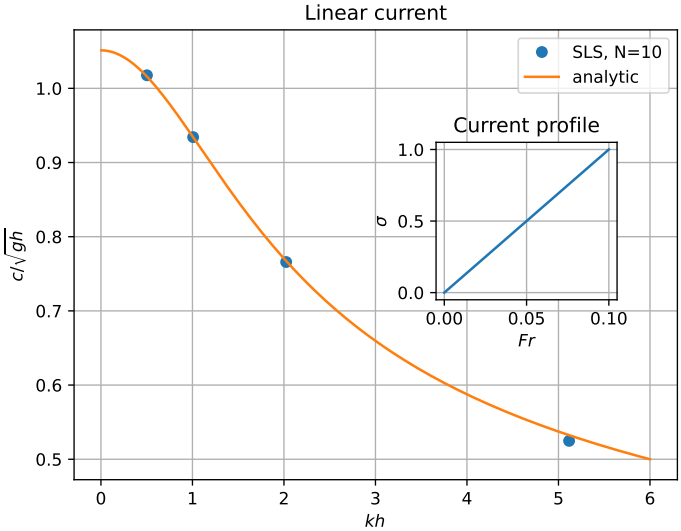


- Wave $\rightarrow$ Amplitude $A = h/100$ and length $\lambda = \{0.5, 1.25, 2.5, 5\}\ \ m$
- Topography $\rightarrow$ $h = 0.4m$
- Current $\rightarrow$ linear: $Fr_{lin} = 0.1\sigma$ and exponential: $Fr_{exp} = 0.1 \exp\left[3(\sigma - 1)\right]$
- Horizontal $\rightarrow$ $N_x = 2000$ cells of equal size.
- Vertical $\rightarrow$ $N_l = 10$ layers with $l_j = 1/N_l$.
- Timestepping $\rightarrow$ $\mathbb{CFL} = 0.9$
- Comparison with approximate dispersion relations [Ellingsen and Li, 2017].

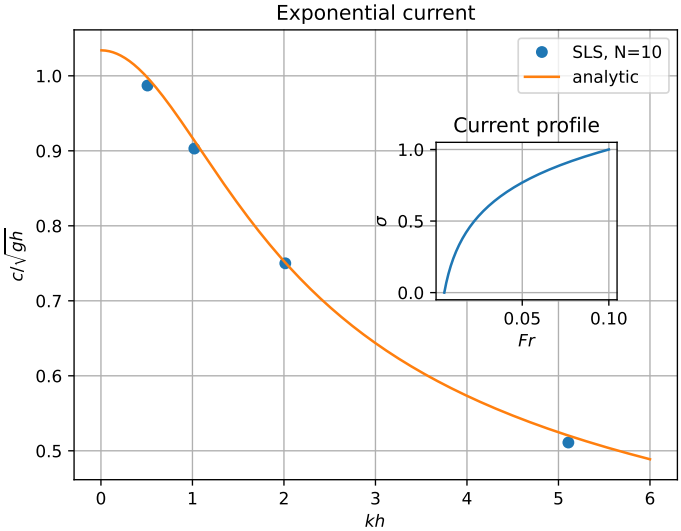
Wave-current interaction



Wave-current interaction



Wave-current interaction



Directions for future research

Possible directions for future research:

- Testing on the complex case of wave+current+topography.
- Add physics like: variable density (stratified flows), effects of viscosity e.t.c.
- Extension to 3D:
  - SWE: 1D→2D
  - VRO: the same
  - PCO/Poisson solver: 2D→3D

Thank you for your attention!

The Free-Surface Euler system

The FSE

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad (12a)$$

$$\frac{Du}{Dt} + \frac{\partial q}{\partial x} + g\frac{\partial \eta}{\partial x} = 0 \qquad (12b)$$

$$\frac{Dw}{Dt} + \frac{\partial q}{\partial z} = 0 \qquad (12c)$$

$$\widetilde{q} = 0 \qquad (13a)$$

$$\frac{\partial \eta}{\partial t} + \widetilde{u}\frac{\partial \eta}{\partial x} - \widetilde{w} = 0 \qquad (13b)$$

$$\widehat{u}\frac{\partial h}{\partial x} + \widehat{w} = 0 \qquad (13c)$$

Where $q \stackrel{\text{def}}{=\!=} \frac{p}{\rho} - g(\eta - z)$ is the dynamic pressure and

$\frac{D(\cdot)}{Dt} \stackrel{\text{def}}{=\!=} \frac{\partial}{\partial t}(\cdot) + u\frac{\partial}{\partial x}(\cdot) + w\frac{\partial}{\partial z}(\cdot)$ the material derivative

The Generalized Coordinate Euler system

Thus, the FSE is rewritten in the domain $(x', \xi) \in \Omega_\xi = \mathbb{R} \times [0, 1]$ as:

The Generalized Coordinate Euler system

$$\frac{\partial}{\partial x'}(Lu) + \frac{\partial}{\partial \xi}(w - uZ_x) = 0 \tag{14a}$$

$$\frac{\partial}{\partial t'}(Lu) + \frac{\partial}{\partial x'}\left(Lu^2 + \frac{1}{2}gL^2 + Lq\right) + \frac{\partial}{\partial \xi}(\vartheta u - Z_x q) = gL\frac{\partial}{\partial x}(L - \eta) \tag{14b}$$

$$\frac{\partial}{\partial t'}(Lw) + \frac{\partial}{\partial x'}(Luw) + \frac{\partial}{\partial \xi}(\vartheta w + q) = 0 \tag{14c}$$

$$\text{BC: } \widetilde{\vartheta} = \widehat{\vartheta} = \widetilde{q} = 0 \tag{14d}$$

Linear dispersion properties

Consider a sine wave of length λ propagating on top of bottom $h = const$ with speed c. Analysis of the linearized model gives:

$$\frac{c^2}{gh} = \frac{2}{kh^2 + 2} \qquad N_l = 1 \tag{15a}$$

$$= \frac{4kh^2 + 32}{kh^4 + 16kh^2 + 32} \qquad N_l = 2 \tag{15b}$$

$$= \frac{6kh^4 + 216kh^4 + 1458}{kh^6 + 54kh^4 + 729kh^2 + 1458} \qquad N_l = 3 \tag{15c}$$

$$= \frac{\sum_{n=0}^{N-1} \alpha_n (kh)^{2n}}{\sum_{k=0}^{N} \beta_k (kh)^{2k}} \qquad N_l = N \tag{15d}$$

with N_l: number of layers and $k = 2\pi/\lambda$: wavenumber.

Boundary data

For wave generation in a presence of a current $V(\sigma) = \{ V_j \ : \ \sigma \in [\sigma_j, \sigma_{j+1}] \}$ we use:

$$L^j_{wc} = h + A \times \cos\left[k(x - ct)\right] \tag{16a}$$

$$(Lu)^j_{wc} = cA \times \ \mathring{u}_j \times \cos\left[k(x - ct)\right] + h V_j \tag{16b}$$

$$(Lw)^j_{wc} = cA \times \mathring{w}_j \times \sin\left[k(x - ct)\right] \tag{16c}$$

$$q^j_{wc} = gA \times \mathring{q}_j \times \cos\left[k(x - ct)\right] \tag{16d}$$

The celerity is approximated by [Ellingsen and Li, 2017]:

$$c = \sqrt{(c_{wave})^2 + (\delta_* - V_{N-1})^2} + \delta_* \tag{17}$$

$$\text{with : } \delta_* = \delta_*[V_j] = \sum_{j=0}^{N_l - 1} V_j (N_{j+1} - N_j) \quad \text{and} \quad N_j = \frac{\sinh(2kh\sigma_j)}{\sinh(2kh)} \tag{18}$$

The multilayered system as an extension of the SWE

The mSWE using the flux and source of the SWE:

$$\frac{\partial U_j}{\partial t} + \frac{\partial}{\partial x} F(U_j) = S\left(U_j; h_j^\circ\right) \qquad \text{with: } U_j = \begin{bmatrix} L_j & L_j u_j & L_j w_j \end{bmatrix}^T \tag{19}$$

$$\text{where: } h_j^\circ \stackrel{\text{def}}{=\!=} h + L_j - H = h + L_j - \sum_{j=0}^{j<N_l} l_j L_j \tag{20}$$

$h_j^\circ \rightarrow$ causes cross layer coupling and <u>loss of hyperbolicity</u> unless:

$$L_j \approx H \iff h_j^\circ \approx h \tag{21}$$

The <u>remeshing</u> technique will ensure that. $\Rightarrow$ The mSWE are solved by consecutive SWE.

The SWE scheme

Consider a tessellation $\bigcup_{i=0}^{N-1}[x_i, x_{i+1}]$ and the Finite Volume discretization:

$$\frac{\partial U_i}{\partial t} + \frac{\widetilde{F}_{i+1/2} - \widetilde{F}_{i-1/2}}{\Delta x_i} = S_i \tag{22}$$

- $\widetilde{F}_{i+1/2} = \widetilde{F}(U^L_{i+1/2}, U^R_{i+1/2}) \rightarrow$ Roe's intercell flux [Roe, 1981]
- Hydrostatic variable reconstruction:

$$\mathbf{U}^{L/R} = \begin{bmatrix} \mathbf{H}^{L/R} \\ \mathbf{P}^{L/R} \\ \mathbf{Q}^{L/R} \end{bmatrix} = \begin{bmatrix} \mathcal{R}^2_{lim}(\mathbf{H} - \mathbf{h}) + \widetilde{\mathbf{h}} \\ \mathcal{R}^2_{lim}(\mathbf{P}) \\ \mathcal{R}^2_{lim}(\mathbf{Q}) \end{bmatrix} \tag{23}$$

- $\mathcal{R}^2_{lim}(\cdot)$: second order (linear) reconstruction operator with superbee limiter.
- $S_i \rightarrow$ approximation of the source term $S(U; h)$:

$$S_i = g(H_i - h_i)\frac{\Delta \widetilde{h}_i}{\Delta x_i} + \frac{1}{2} g \frac{\Delta(\widetilde{h}^2)}{\Delta x_i}$$

The VRO

Define the remeshing $L^0 \mapsto L^*$ and integrate over $[t, t+\Delta t] \times [\xi_j, \xi_{j+1}]$ to obtain:

$$\vartheta^*_{j+1} = \begin{cases} \vartheta^*_j - \dfrac{l_j}{\Delta t}\left(L^*_j - L^0_j\right) & , \ j \geq 0 \\ 0 & , \ j = -1 \end{cases} \tag{24}$$

where $\quad \vartheta^*_j \overset{\text{def}}{=\!=} \frac{1}{\Delta t}\int_t^{t+\Delta t} \vartheta_j dt \quad$ and $\quad \vartheta_j = \vartheta_{\xi=\xi_j}$.
Velocity update with explicit Finite Volume scheme:

$$\mathcal{U}^*_j = \mathcal{U}^0_j - \frac{\Delta t}{l_j}\left[\widetilde{G}_{j+1/2} - \widetilde{G}_{j-1/2}\right] \tag{25}$$

with $\mathcal{U} = [Lu, Lw]^T$

- $\widetilde{G}_{i+1/2} = \widetilde{G}\left(\mathcal{U}_j, \mathcal{U}_{j+1}; \vartheta^*_{j+1}\right) \longrightarrow$ Rusanov numerical flux.
- $\vartheta_0 = \vartheta_{N_l} = 0 \Rightarrow \widetilde{G}_{-1/2} = \widetilde{G}_{N_l-1/2} = 0 \longrightarrow$ no further B.C. treatment needed.

The Pressure Correction Operator

The projection method [Chorin, 1968] in a time window $[t, t+\Delta t]$ in the cartesian frame:

$$\left\{\begin{array}{c}\dfrac{\mathbf{u}-\mathbf{u}^*}{\Delta t}+\nabla q=0\\ \nabla\cdot\mathbf{u}=0\end{array}\right\}\Longleftrightarrow\left\{\begin{array}{c}\mathbf{u}=\mathbf{u}^*-\Delta t\,(\nabla q)\\ \nabla^2 q=\dfrac{\nabla\cdot\mathbf{u}^*}{\Delta t}\end{array}\right\} \tag{26}$$

The Poisson equation on the (x, ξ) system has two difficulties:

1. Transformation of derivatives $\Rightarrow$ Large computational stencil
2. Enforcement of the bottom B.C.

A partially implicit method [Liu et al., 2016] is used to bypass them.

Partially Implicit approach

Explicit:

$$P' = P^* + Z_x \frac{\partial \lambda_{\approx}}{\partial \xi} \tag{27a}$$

$$Q' = Q^* + Z_x L \frac{\partial \lambda_{\approx}}{\partial x} \tag{27b}$$

Implicit:

$$P = P' - L \frac{\partial \lambda}{\partial x} \tag{28a}$$

$$\Psi = \Psi' - \frac{\partial \lambda}{\partial \xi} \tag{28b}$$

$$L \frac{\partial P}{\partial x} + \frac{\partial \Psi}{\partial \xi} = 0 \tag{28c}$$

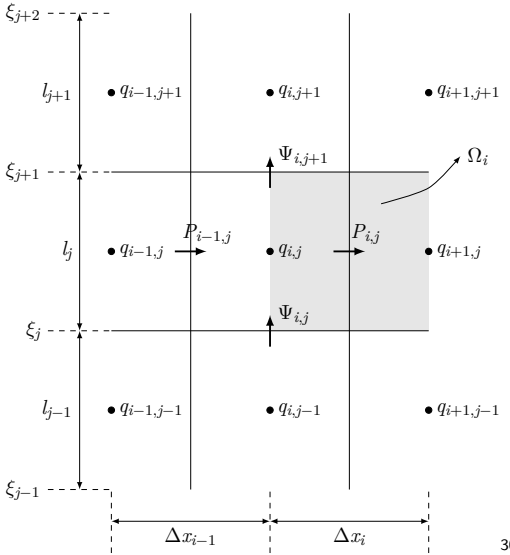
$$\text{BC: } \lambda_{\xi=1} = \Psi_{\xi=0} = 0 \tag{28d}$$

with $P \stackrel{\text{def}}{=} Lu$, $Q \stackrel{\text{def}}{=} Lw$, $\Psi \stackrel{\text{def}}{=} Q - Z_x P$ and $\lambda \stackrel{\text{def}}{=} \Delta t \cdot q$

$q_{\approx}$: approximate field calculated at the previous timestep.

Grid staggering

- Staggered C-grid [Arakawa and Lamb, 1977]
- Ψ_{ij} are shifted to the nodes (and back) by interpolation.
- Finite differences $\rightarrow$ 5-diagonal system $[\mathbf{PCO}]\,\mathbf{q} = \mathbf{div}$.



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