Supplementary Material
Taxonomy of Cliff Failure Criteria
Phase Field Modelling and Parallels with Other Models

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Standard ice shelf models

- Ice assumed to be Maxwell viscoelastic material

$$\sigma = \sigma_e = \sigma_v$$
$$\varepsilon = \varepsilon_e + \varepsilon_v$$

- Viscosity is strain rate dependent - Glen-Nye flow law/Norton material

$$\eta = \eta(\dot{\varepsilon}_v) = \frac{1}{2} A(T')^{-1/n} |\dot{\varepsilon}_v|^{1/n-1}$$

- Long time scale $\rightsquigarrow$ modelled as purely viscous flow - Vectorial p-Stokes system

$$-\nabla \cdot (-p\mathrm{1} + 2\eta(\nabla_s \mathrm{v})\nabla_s \mathrm{v}) = f$$
$$\nabla \cdot \mathrm{v} = 0$$

- Vertical direction usually "ignored" $\rightsquigarrow$ Shallow shelf approximation

$$\frac{\partial v_x}{\partial z} \approx 0, \quad \frac{\partial v_y}{\partial z} \approx 0$$

Maxwell VE equations

- Maxwell viscoelastic material:

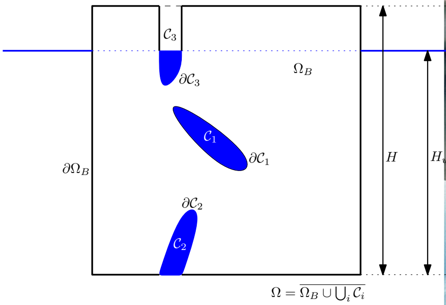
$$\sigma = \sigma_e = \sigma_v$$
$$\varepsilon = \varepsilon_e + \varepsilon_v$$

- Rheology:

$$\sigma_e = \lambda\,\mathrm{tr}(\varepsilon_e)\mathbb{1} + 2\mu\varepsilon_e$$
$$\sigma_v = -p\mathbb{1} + 2\eta(\dot{\varepsilon}_v)\dot{\varepsilon}_v$$

- Glen-Nye flow law:

$$\eta(\dot{\varepsilon}_v) = \frac{1}{2}A(T')^{-1/n}|\dot{\varepsilon}_v|^{1/n-1}$$



Maxwell VE equations

- Maxwell viscoelastic material:

$$\sigma = \sigma_e = \sigma_v$$

$$\varepsilon = \varepsilon_e + \varepsilon_v$$

- Free energy function:

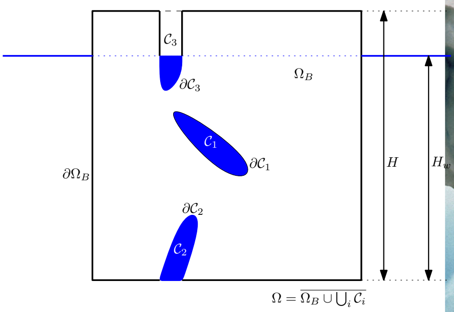
$$\psi_e(\varepsilon_e) = \frac{\lambda}{2}\,\mathrm{tr}(\varepsilon_e)^2 + \mu\,\mathrm{tr}(\varepsilon_e^2)$$

- Dissipation function:

$$\phi_v(\dot{\varepsilon}_v) = 2\eta(\mathrm{dev}(\dot{\varepsilon}_v))\,\mathrm{dev}(\dot{\varepsilon}_v) : \mathrm{dev}(\dot{\varepsilon}_v)$$

- Small strain, small displacement:

$$\varepsilon = \nabla_s \mathrm{u} = \nabla_s \mathrm{v} + \nabla_s \mathrm{w}$$



Maxwell VE equations

From energy balance:

$$
\begin{aligned}
0 = &\int_{\Omega_B} [\delta_u \psi_e + \delta_{\dot{u}} \phi - \mathsf{f}] \cdot \dot{\mathsf{u}} + [\delta_w \psi_e + \delta_{\dot{w}} \phi] \cdot \dot{\mathsf{w}} \, \mathrm{d}\, V \\
&+ \int_{\partial_N \Omega_B} [(\partial_{\nabla_s u} \psi_e + \partial_{\nabla_s \dot{u}} \phi) \cdot \mathsf{n} - \mathsf{t}] \cdot \dot{\mathsf{u}} \, \mathrm{d}\, S + \int_{\partial_N \Omega_B} [(\partial_{\nabla_s w} \psi_e + \partial_{\nabla_s \dot{w}} \phi) \cdot \mathsf{n}] \cdot \dot{\mathsf{w}} \, \mathrm{d}\, S
\end{aligned}
$$

To strong form:

$$
\begin{aligned}
-\nabla \cdot \left(-p\mathbb{1} + 2\mu \,\mathrm{dev}\left(\nabla_s\left(\mathsf{u} - \mathsf{w}\right)\right)\right) &= \mathsf{f} && \text{in } \Omega_B \\
2\eta(\mathrm{dev}(\nabla_s \dot{\mathsf{w}}))\,\mathrm{dev}(\dot{\mathsf{w}}) - 2\mu \,\mathrm{dev}\left(\nabla_s\left(\mathsf{u} - \mathsf{w}\right)\right) &= 0 && \text{in } \Omega_B \\
\nabla \cdot (\mathsf{u} - \mathsf{w}) + \left(\lambda + \frac{2\mu}{3}\right)^{-1} p &= 0 && \text{in } \Omega_B \\
\nabla \cdot \dot{\mathsf{w}} &= 0 && \text{in } \Omega_B \\
\left(\lambda \nabla \cdot (\mathsf{u} - \mathsf{w})\,\mathbb{1} + 2\mu \nabla_s\left(\mathsf{u} - \mathsf{w}\right)\right) \cdot \mathsf{n} &= \mathsf{t} && \text{on } \partial_E \Omega_B \cup \partial_N \Omega_B.
\end{aligned}
$$

Variational formulation of fracture

Energy balance:

$$\Pi(\mathbf{u}, \mathcal{C}) = \Pi_{int}(\mathbf{u}, \Omega \setminus \mathcal{C}) - \int_{\Omega \setminus \mathcal{C}} \mathbf{f} \cdot \dot{\mathbf{u}} \, \mathrm{d}V - \int_{\partial_N \Omega} \mathbf{t} \cdot \dot{\mathbf{u}} \, \mathrm{d}S + G_c \int_{\mathcal{C}} k(\mathbf{x}) \, \mathrm{d}\mathcal{H}^{N-1}.$$

Minimisation problem:

$$\inf \Pi(\mathbf{u}, \mathcal{C})$$
$$+ \ \textit{crack irreversibility condition}$$

ISSUE: Numerical implementation is challenging and requires carefully tailored techniques!

Variational formulation of fracture

Energy balance:

$$\Pi(\mathrm{u}, \mathcal{C}) = \Pi_{int}(\mathrm{u}, \Omega \setminus \mathcal{C}) - \int_{\Omega \setminus \mathcal{C}} \mathrm{f} \cdot \dot{\mathrm{u}} \, \mathrm{d} V - \int_{\partial_N \Omega} \mathrm{t} \cdot \dot{\mathrm{u}} \, \mathrm{d} S + G_c \int_{\mathcal{C}} k(\mathrm{x}) \, \mathrm{d} \mathcal{H}^{N-1}.$$

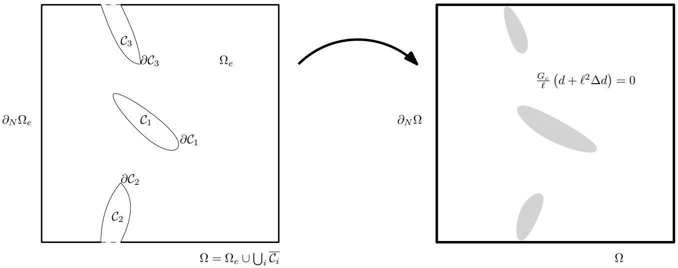
Minimisation problem:

$$\inf \Pi(\mathrm{u}, \mathcal{C})$$

$+$ *crack irreversibility condition*

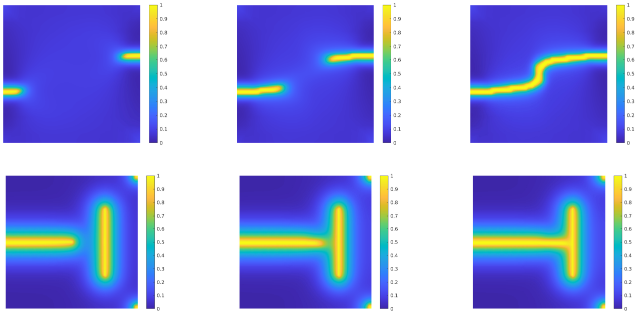
IDEA: Regularise the "nasty" stuff by an elliptic functional [Ambrosio, Tortorelli 1990-2].

Why phase field?



- Popularised by Fix, Langer, Bourdin, Francfort, Marigo, Miehe, ...
- More recently for ice by Clayton, Duddu, Martinez-Pañeda, Shen, Waisman, Guo, ...
- Elimination of explicit treatment of boundary conditions and evolving geometry
- Replaces them with additional differential equations
- Reproduces curvilinear cracks, kinks, branching, arrest, and crack nucleation.

Why phase field?



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Phase field regularisation

$$d - \ell^2 \Delta d = 0 \text{ in } \Omega, \qquad \nabla d \cdot n = 0 \text{ on } \partial\Omega, \qquad d = 1 \text{ on } \Gamma$$

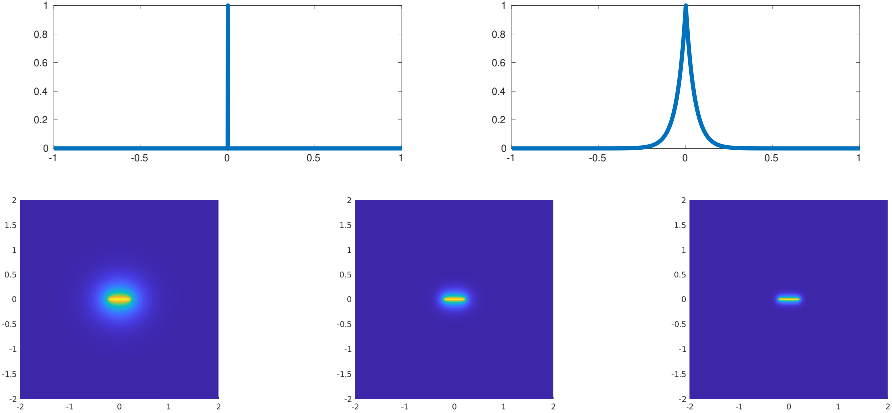

Figure: Regularisation of a crack $\Gamma = [-0.2, 0.2] \times 0$ for parameters $\ell = 0.25, 0.10, 0.05$

Variational formulation of fracture

Energy balance:

$$\Pi(\mathrm{u}, \mathcal{C}) = \Pi_{int}(\mathrm{u}, \Omega \setminus \mathcal{C}) - \int_{\Omega \setminus \mathcal{C}} \mathrm{f} \cdot \dot{\mathrm{u}} \,\mathrm{d}\, V - \int_{\partial_N \Omega} \mathrm{t} \cdot \dot{\mathrm{u}} \,\mathrm{d}\, S + G_c \int_{\mathcal{C}} k(\mathrm{x}) \,\mathrm{d}\, \mathcal{H}^{N-1}$$

Minimisation problem:

$$\inf \Pi(\mathrm{u}, \mathcal{C})$$
$$+ \;\; \textit{crack irreversibility condition}$$

Regularised energy balance:

$$\Pi_\ell(\mathrm{u}, d) = \Pi_{int}(\mathrm{u}, d, \Omega) - \int_{\Omega} g(d) \mathrm{f} \cdot \dot{\mathrm{u}} \,\mathrm{d}\, V - \int_{\partial_N \Omega} g(d) \mathrm{t} \cdot \dot{\mathrm{u}} \,\mathrm{d}\, S$$
$$+ G_c \int_{\Omega} \frac{1}{2\ell} \left(d^2 + \ell^2 |\nabla d|^2 \right) \mathrm{d}\, V$$

Degradation of functions

We want a degradation function $g(d)$ so that

$$g(0) = 1, \quad g(1) = 0, \quad g'(1) = 0.$$

One possibility:

$$g(d) = (1 - d)^2.$$

Q: How do we degrade free energy and degradation functions?

$$\text{(??)} \quad \widetilde{\psi}_e(\varepsilon_e, d) = g(d) \left(\frac{\lambda}{2} \operatorname{tr}(\varepsilon_e)^2 + \mu \varepsilon_e : \varepsilon_e \right) \quad \text{(??)}$$

ISSUE: Behaves the same under tension and compression.

Degradation of functions

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One possibility:

$$g(d) = (1 - d)^2.$$

Q: How do we degrade free energy and degradation functions?

Free energy function:

$$\widetilde{\psi}_e(\varepsilon_e, d) = g(d)\left(\psi_e^+(\varepsilon_e) - \psi_c\right) + \left(\psi_e^-(\varepsilon_e) + \psi_c\right)$$
$$\psi_e^\pm = (\lambda/2 + \mu/3)\langle \mathrm{tr}(\varepsilon_e)\rangle_\pm^2 + \mu\, \mathrm{tr}\left(\mathrm{dev}(\varepsilon_{e,\pm})^2\right)$$

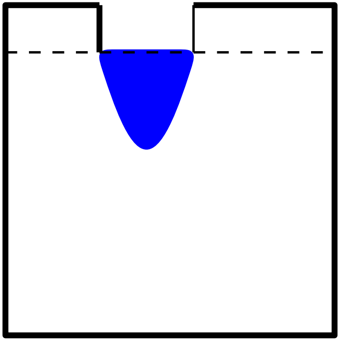
Dissipation function:

$$\widetilde{\phi}(\dot{\varepsilon}_v, d) = g(d) 2\eta(\mathrm{dev}(\dot{\varepsilon}_v))\, \mathrm{dev}(\dot{\varepsilon}_v) : \mathrm{dev}(\dot{\varepsilon}_v).$$

Hydrofracture

Inclusion of pressurised cracks:

$$\int_{\cup_i \partial \mathcal{C}_i} p_w \mathsf{n}\cdot\dot{\mathsf{u}}\,\mathrm{d}\,S = \int_{\Omega_B} \nabla\cdot(p_w\dot{\mathsf{u}})\,\mathrm{d}\,V - \int_{\partial_E\Omega_B} p_w \mathsf{n}\cdot\dot{\mathsf{u}}\,\mathrm{d}\,S.$$



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Inclusion of pressurised cracks:

$$\int_{\cup_i \partial \mathcal{C}_i} p_w \mathsf{n} \cdot \dot{\mathsf{u}} \,\mathrm{d}\, S = \int_{\Omega_B} \nabla \cdot (p_w \dot{\mathsf{u}}) \,\mathrm{d}\, V - \int_{\partial_E \Omega_B} p_w \mathsf{n} \cdot \dot{\mathsf{u}} \,\mathrm{d}\, S.$$

Extension to Ω:

$$\int_{\Omega_B} \nabla \cdot (p_w \dot{\mathsf{u}}) \,\mathrm{d}\, V \approx \int_{\Omega} g(d) \nabla \cdot (p_w \dot{\mathsf{u}}) \,\mathrm{d}\, V$$

$$\int_{\partial_E \Omega_B} p_w \mathsf{n} \cdot \dot{\mathsf{u}} \,\mathrm{d}\, S \approx \int_{\partial_N \Omega} g(d) p_w \mathsf{n} \cdot \dot{\mathsf{u}} \,\mathrm{d}\, S.$$

Hydrofracture

Inclusion of pressurised cracks:

$$\int_{\cup_i \partial \mathcal{C}_i} p_w \mathsf{n} \cdot \dot{\mathsf{u}} \, \mathrm{d}\, S = \int_{\Omega_B} \nabla \cdot (p_w \dot{\mathsf{u}}) \, \mathrm{d}\, V - \int_{\partial_E \Omega_B} p_w \mathsf{n} \cdot \dot{\mathsf{u}} \, \mathrm{d}\, S.$$

Thanks to Divergence Theorem:

$$\int_{\Omega_B} \nabla \cdot (p_w \dot{\mathsf{u}}) \, \mathrm{d}\, V \approx - \int_{\Omega} p_w \nabla g(d) \cdot \dot{\mathsf{u}} \, \mathrm{d}\, V + \int_{\partial_N \Omega} g(d) p_w \mathsf{n} \cdot \dot{\mathsf{u}} \, \mathrm{d}\, S$$

$$\int_{\partial_E \Omega_B} p_w \mathsf{n} \cdot \dot{\mathsf{u}} \, \mathrm{d}\, S \approx \int_{\partial_N \Omega} g(d) p_w \mathsf{n} \cdot \dot{\mathsf{u}} \, \mathrm{d}\, S.$$

Hydrofracture

Inclusion of pressurised cracks:

$$\int_{\cup_i \partial \mathcal{C}_i} p_w \mathrm{n} \cdot \dot{\mathrm{u}} \,\mathrm{d}\, S = \int_{\Omega_B} \nabla \cdot (p_w \dot{\mathrm{u}}) \,\mathrm{d}\, V - \int_{\partial_E \Omega_B} p_w \mathrm{n} \cdot \dot{\mathrm{u}} \,\mathrm{d}\, S.$$

Combined:

$$\int_{\cup_i \partial \mathcal{C}_i} p_w \mathrm{n} \cdot \dot{\mathrm{u}} \,\mathrm{d}\, S \approx \int_{\Omega} 2 p_w (1 - d) \nabla d \cdot \dot{\mathrm{u}} \,\mathrm{d}\, V.$$

Intuition:

- Term $(1 - d)$ acts like an indicator function.
- Term ∇d acts like the unit normal vector.

Maxwell viscoelastic fracture model

Strong form:

$$
\begin{aligned}
-\nabla \cdot \left(g(d)\sigma_e^+ + \sigma_e^-\right) &= [g(d)\mathsf{f} - p_w \nabla g(d)] && \text{in } \Omega \\
g(d) 2\eta(\operatorname{dev}(\nabla_s \dot{\mathsf{w}}))\operatorname{dev}(\dot{\mathsf{w}}) &= \left[2g(d)\operatorname{dev}(\sigma_e^+) - 2\operatorname{dev}(\sigma_e^-)\right] && \text{in } \Omega \\
\nabla \cdot (\mathsf{u} - \mathsf{w}) + \mu\left(\lambda + \frac{2\mu}{3}\right)^{-1} p &= 0 && \text{in } \Omega \\
\nabla \cdot \dot{\mathsf{w}} &= 0 && \text{in } \Omega \\
\left(g(d)\sigma_e^+ + \sigma_e^-\right) \cdot \mathsf{n} &= g(d)\mathsf{t} && \text{on } \partial_N \Omega \\
d - \ell^2 \Delta d &= 2C\ell(1-d)\mathcal{D} && \text{in } \Omega \\
\nabla d \cdot \mathsf{n} &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Stresses: $\sigma_e^\pm := \left(\langle -p\rangle_\pm \mathbb{1} + 2\operatorname{dev}\left(\nabla_s\left(\mathsf{u} - \mathsf{w}\right)_\pm\right)\right)$

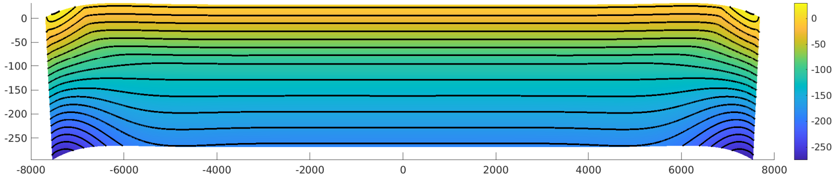
Fracture forcing: $\mathcal{D} = \max_{s\in[0,t]} \langle \psi_e^+(\varepsilon_e(x,s)) - \psi_c \rangle_+$

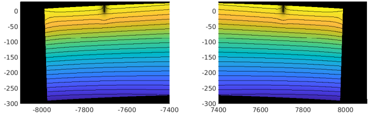
Numerical setup

- Mixed finite element solver (2D/3D) - KRAKEN
- Updated Lagrangian formulation
- Alternating minimisation scheme married to inexact Newton to bypass nonconvexity.

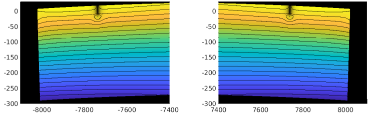
Results

$H = 300m$

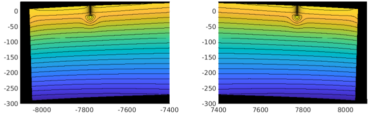




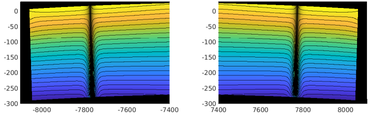
(A) $t = 0$ days



(B) $t = 98$ days

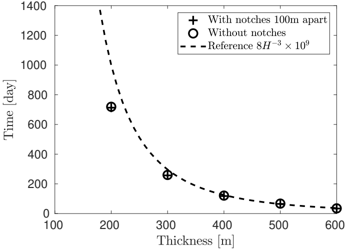


(C) $t = 258$ days

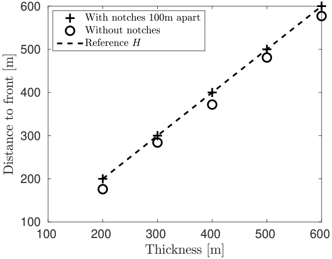


(D) $t = 260$ days

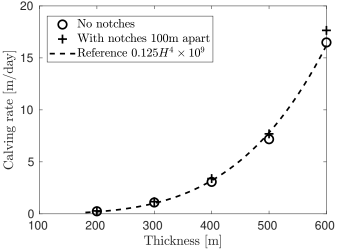
RESULTS



With notches 100m apart
Without notches
Reference $8H^{-3} \times 10^9$
Time [day]
1400
1200
1000
800
600
400
200
0
100
200
300
400
500
600
Thickness [m]



With notches 100m apart
Without notches
Reference H
Distance to front [m]
600
500
400
300
200
100
100
200
300
400
500
600
Thickness [m]



No notches
With notches 100m apart
Reference $0.125H^4 \times 10^9$
Calving rate [m/day]
20
15
10
5
0
100
200
300
400
500
600
Thickness [m]