The effect of linear dispersive errors on nonlinear time-stepping accuracy in weather and climate models

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1. Introduction

The next generation of weather and climate models will need to run increasingly complex simulations whilst meeting a wall-clock time constraint. A way to achieve this, without excessive computational power, is the use of a larger time-step, Δt. However, this is challenging to implement in time-stepping algorithms. **Explicit methods** have stability time-step limits from the fastest wave oscillations. **Implicit methods** have increased stability regions, but lose accuracy with larger time-steps.

The presence of multiple time-scales increases the numerical complexity of weather and climate models. Additionally, the nonlinear terms introduce slow phase shifts that can be missed with a large Δt. Figure 1 demonstrates this for the Rotating Shallow Water Equations (RSWEs); a Gaussian height perturbation disperses and reforms more times in the nonlinear system.

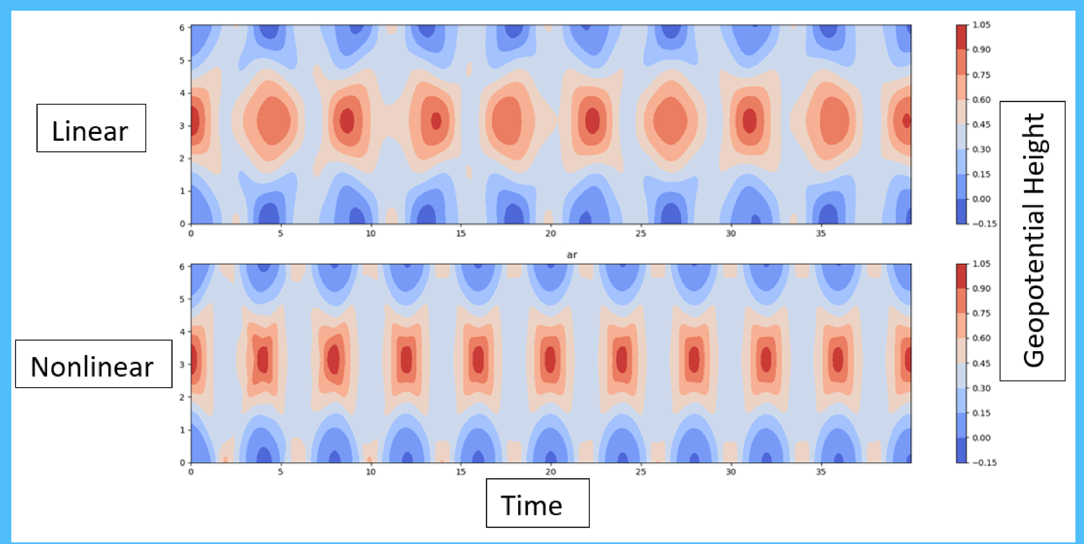

Fig. 1: A 1D Gaussian perturbation in the RSWEs. The nonlinear case (bottom) has ten reformations, whereas the linear (top) only has nine.

The standard formulation of weather and climate-type PDEs is,

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\epsilon} L\mathbf{u} = \mathcal{N}(\mathbf{u}), \quad \epsilon \in \mathbb{R}, \quad \epsilon \neq 0 \tag{1}$$

We consider an L which generates purely oscillatory linear behaviour, with frequencies on a $\mathcal{O}(\epsilon)$ time-scale. When ϵ is small, we expect fast oscillations; when $\epsilon \sim \mathcal{O}(1)$ the oscillations are slower. A small ϵ, as is present in many weather and climate applications, leads to strict explicit time-step limits.

We present two types of analysis. An **analytical analysis** of a nonlinear 'triadic error' provides comparisons of time-steppers based on their stability polynomials. Test cases enable a **numerical analysis** of any time-stepper in a given model. Three models are examined numerically: A pseudospectral model, Gusto (compatible finite elements), and LFRic (MetOffice).

2. The RSWEs:

We consider a rotating fluid in a doubly periodic domain, with no topography,

$$\frac{D\mathbf{u}}{Dt} + f\hat{\mathbf{k}} \times \mathbf{u} + g\nabla\eta = 0 \tag{2}$$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \mathbf{u}) + H_0(\nabla \cdot \mathbf{u}) = 0 \tag{3}$$

where g is the gravitational force, H_0 is the mean height, and f is the constant value of rotation [Val17]. There are three dispersion relation branches, $\alpha \in \{-1, 0, +1\}$, corresponding to fast (± 1) and slow (0) modes,

$$\omega_{\mathbf{k}}^{\alpha} = \alpha \psi_{\mathbf{k}}, \quad \psi_{\mathbf{k}} = \sqrt{f^2 + \gamma^2 |\mathbf{k}|^2} = \frac{1}{\epsilon}\sqrt{1 + |\mathbf{k}|^2}, \quad |\mathbf{k}| = \sqrt{k^2 + l^2} \tag{4}$$

3. Analytical Analysis: The Triadic Error

We map the standard form of Eq. (1) to a space without $\mathcal{O}(\epsilon)$ oscillations [Sch92; EM96]:

$$\mathbf{u}(x, y, t) = e^{-t\mathcal{L}} \mathbf{v}(x, y, t) \tag{5}$$

$$\frac{\partial \mathbf{v}}{\partial t} = e^{t\mathcal{L}} \mathcal{N}(e^{-t\mathcal{L}}\mathbf{v}, e^{-t\mathcal{L}}\mathbf{v}) \tag{6}$$

In this alternative basis, we can find an evolution equation for spectral coefficients that only has contributions from **triads**; these are sets of three waves that satisfy $\mathbf{k} = \mathbf{k}_a + \mathbf{k}_b$. We identify the temporal evolution of these triads through the **triadic propagator**, T,

$$T(\Omega, t) = e^{i\Omega^{\alpha,\alpha_a,\alpha_b}_{\mathbf{k},\mathbf{k}_a,\mathbf{k}_b} t}, \quad \Omega^{\alpha,\alpha_a,\alpha_b}_{\mathbf{k},\mathbf{k}a,\mathbf{k}b} = \omega^{\alpha_a}_{\mathbf{k}_a} + \omega^{\alpha_b}_{\mathbf{k}_b} - \omega^{\alpha}_{\mathbf{k}} \tag{7}$$

where the **triadic frequency**, Ω, is a function of the three wavenumbers and modes (α).

- **Direct resonances** occur when $\Omega = 0$. These construct long-time dynamics and are the only remaining interactions when $\epsilon \to 0$ [EM96].
- **Near resonances** have a small, but non-zero, Ω. For finite values of ϵ, like in Eq. (1), these have an important contribution [New69; SW01].
- We analyse the dominant subset of triads over a certain time-scale, by computing errors for all interactions satisfying $|\Omega| \leq \Omega_C$.

We now consider the oscillatory Dahlquist test equation [Dur10]:

$$\frac{d\mathbf{u}}{dt} = i\omega\mathbf{u} \to \mathbf{u}(t + \Delta t) = e^{i\omega\Delta t}\mathbf{u}(t) \tag{8}$$

A time-stepper's approximation to $e^{i\omega\Delta t}$ is given by its stability polynomial, $P(i\omega\Delta t)$. Combining each contribution in the triad obtains the numerical representation, T_N,

$$T_N(\Omega, \Delta t) = P(i\omega^{\alpha_a}_{\mathbf{k_a}}\Delta t) P(i\omega^{\alpha_b}_{\mathbf{k_b}}\Delta t) P(-i\omega^{\alpha}_{\mathbf{k}}\Delta t) \tag{9}$$

The **triadic error**, E, is the difference over one time-step between this approximation of the triadic propagator and the true expression: $E(\Omega, \Delta t) = \|T(\Omega, \Delta t) - T_N(\Omega, \Delta t)\|$.

In the RSWEs, we compare five time-stepping methods—RK4, Crank-Nicolson, AB3, TR-BDF2, and ETD-RK2—for differing Ω_C. $\Omega_C = 0.1$ only contains direct resonances for our example discretisation (32 by 32). $\Omega_C = 5$ contains a large number of near-resonances.

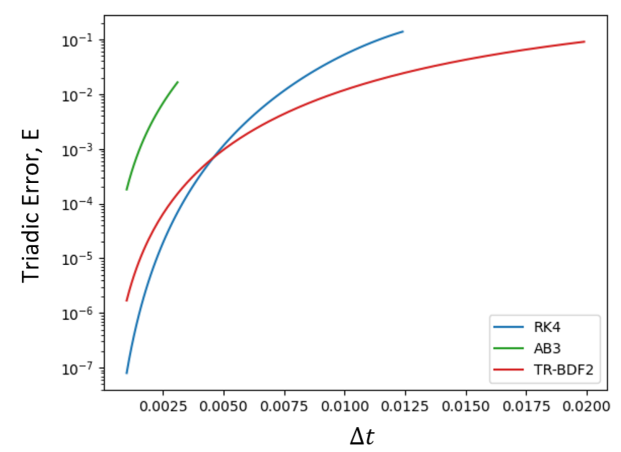

Fig. 2: Comparisons at $\Omega_C = 0.1$

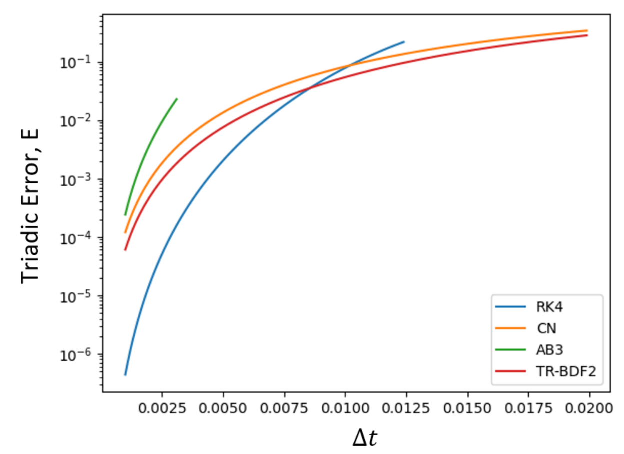

Fig. 3: Comparisons at $\Omega_C = 5$

4. Numerical Analysis: Test Cases

Test Case 1: Gaussian ICs

The first test case contains a Gaussian height perturbation, which disperses and reforms over the entire simulation. Large time-steps can induce phase errors in the form of incorrect reformation times of the Gaussian (Fig. 4). This is diagnosed by computing a spectral error, which quantifies the accuracy of the kinetic energy exchange (Fig. 5).

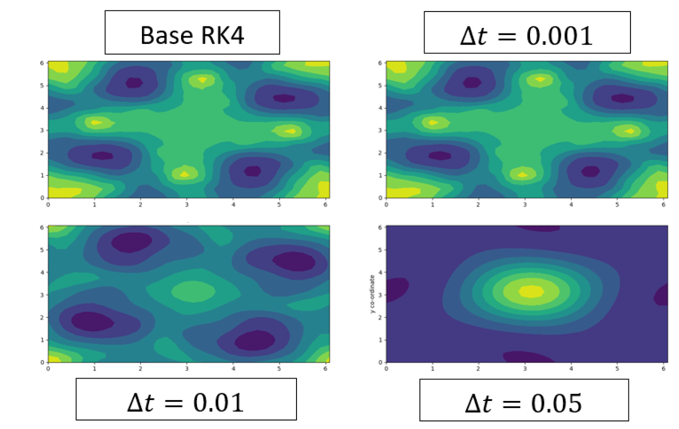

Fig. 4: The pseudospectral Gaussian height field at $t = 20$, using large-time steps with TR-BDF2.

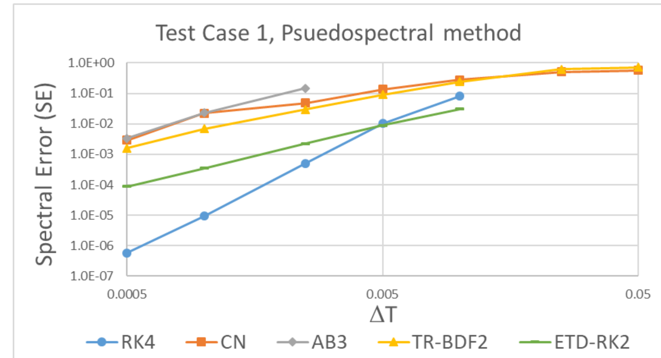

Fig. 5: Spectral errors for the five time-steppers in Section 3, for the pseudospectral simulations.

Test Case 2: Triadic ICs

These initialise linear waves, of specific wavenumbers and modes (α), to excite certain triads.

Test Case 2a only initialises one fast wave and one slow wave. A third fast wave will form to complete a directly resonant triad.

Test Case 2b initialises two fast waves of equal wavenumber and several slow waves. This enables a redistribution of the fast wave energy into rings in wavenumber space (Figure 6).

A height field difference, relative to a fine time-step solution, is used to quantify phase errors.

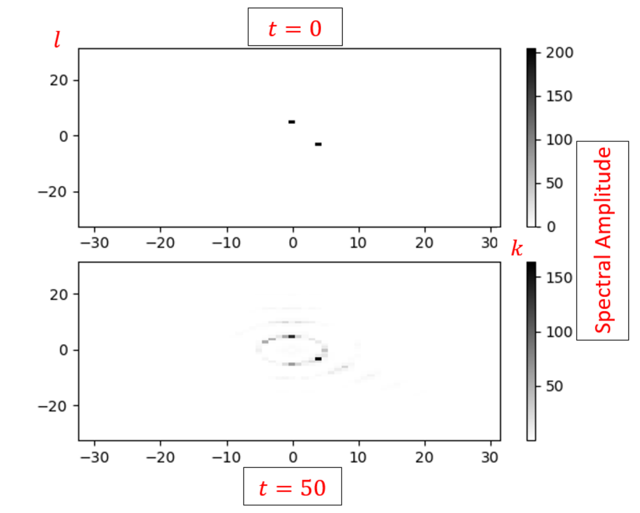

Fig. 6: Fast mode energy in wavenumber space, for Test Case 2b, at the initial and final states.

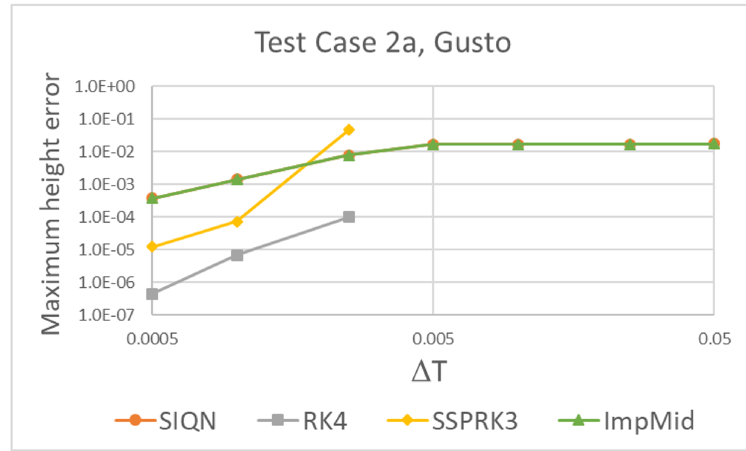

Fig. 7: Test Case 2a with Gusto (Compatible FEM)

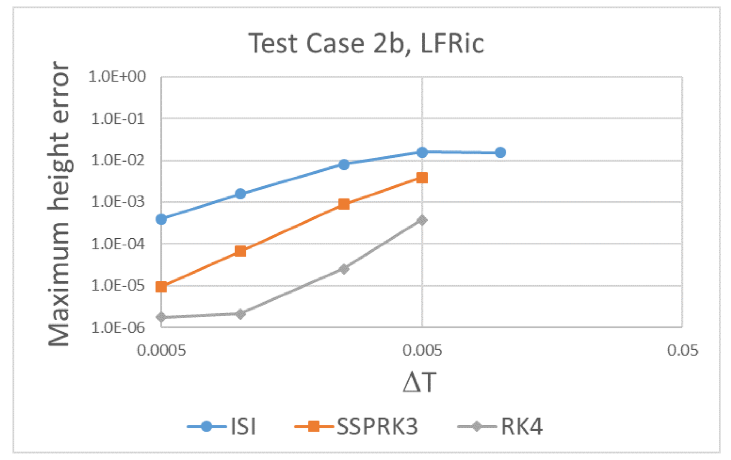

Fig. 8: Test Case 2b with LFRic (MetOffice)

Analytical Conclusions

- Higher order Runge-Kutta (RK) schemes have increased triadic accuracy. RK4 performs relatively better than TR-BDF2 with a larger number of near-resonant triads.
- CN has zero error for direct resonances. It generates increasingly larger errors than TR-BDF2 as more near-resonances are considered.
- AB3 has relatively large errors, even when disregarding its computational modes.
- ETD schemes commit zero triadic error, as use an exact form of the linear terms.

Numerical Conclusions

- Test case 1 shows how the increased stability of implicit methods often comes at the cost of larger phase errors in the nonlinear dynamics.
- In the pseudospectral model, ETD-RK2 performs very well for a second-order scheme.
- In Gusto and LFRic, RK4 performs the best, then SSPRK3, at stable time-steps.
- For test case 2, the implicit methods already have completely out-of-phase solutions with moderate time-steps. Larger time-steps increase the frequency of this phase error.

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