The dipole–multipole transition in planetary dynamos (Supplementary Material)

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Governing equations of numerical simulation

We consider an electrically conducting fluid between two concentric, corotating spherical surfaces that correspond to the inner core boundary (ICB) and the core-mantle boundary (CMB).The ratio of inner to outer radius r_i/r_o is chosen to be 0.35. In the Boussinesq approximation, the non-dimensional MHD equations for the velocity, magnetic field and temperature are given by,

$$\begin{aligned} E\ Pm^{-1}\left[\frac{\partial \boldsymbol{u}}{\partial t}+(\nabla\times\boldsymbol{u})\times\boldsymbol{u}\right]+\hat{\boldsymbol{z}}\times\boldsymbol{u} &= -\nabla p^{*}+RaqT\boldsymbol{r} \\ &\quad +(\nabla\times\boldsymbol{B})\times\boldsymbol{B}+E\nabla^{2}\boldsymbol{u} \end{aligned} \tag{1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})+\nabla^{2}\boldsymbol{B} \tag{2}$$

$$\frac{\partial T}{\partial t}+\boldsymbol{u}.\nabla T=q\nabla^{2}T \tag{3}$$

$$\nabla.\boldsymbol{u}=\nabla.\boldsymbol{B}=0 \tag{4}$$

Scaling and dimensionless numbers are given in the next slide.

Scaling and Dimensionless numbers

Quantity	Scaling
Lengths	Shell thickness (L)
Velocity (u)	η/L
Time	L^2/η
Magnetic field (B)	$(2\Omega\rho\mu\eta)^{1/2}$

Parameter	Definition
Pr (Prandtl number)	ν/κ
Pm (Magnetic Prandtl number)	ν/η
E (Ekman number)	$\nu/2\Omega L^2$
Λ (Elsasser)	$B_0/\sqrt{2\Omega\mu\rho\eta}$
Ra (Modified Rayleigh number)	$g\alpha\Delta TL^2/2\Omega\kappa$
Ra_ℓ (local Rayleigh number)	$g\alpha\Delta T\delta^2/2\Omega\kappa$
q (Roberts number)	Pm/Pr

here,

Rotation rate	Ω	Fluid mean density	ρ
Magnetic permeability	μ	Magnetic diffusivity	η
Gravitational acceleration	g	Kinematic viscosity	ν
Thermal diffusivity	κ	Coefficient of thermal expansion	α

Fundamental frequencies and MAC waves

The dimensional frequencies ω_M^2, $-\omega_A^2$ and ω_C^2 in the dynamo are given by

$$\omega_M^2 = \frac{(\boldsymbol{B}\cdot\boldsymbol{k})^2}{\mu\rho}, \quad \omega_A^2 = g\alpha\beta\left(\frac{k_z^2+k_\phi^2}{k^2}\right), \quad \omega_C^2 = \frac{4(\boldsymbol{\Omega}\cdot\boldsymbol{k})^2}{k^2}, \tag{5}$$

Where, β is negative for unstable stratification.
and scaling the frequencies by η/L^2, we obtain in dimensionless units,

$$\omega_M^2 = \frac{Pm}{E}(\boldsymbol{B}\cdot\boldsymbol{k})^2, \quad \omega_A^2 = \frac{Pm^2\,Ra}{Pr\,E}\left(\frac{{k_z}^2+k_\phi^2}{k^2}\right), \quad \omega_C^2 = \frac{Pm^2}{E^2}\frac{k_z^2}{k^2}, \tag{6}$$

where k_s, k_ϕ and k_z are the radial, azimuthal and axial wavenumbers in cylindrical coordinates (s,ϕ,z), $k_\phi = m/s$, where m is the spherical harmonic degree, and $k^2 = k_s^2 + k_\phi^2 + k_z^2$.
Here, ω_M^2, ω_A^2, ω_C^2 represent the squares of the frequencies of Alfvèn, buoyancy and linear inertial waves respectively.
By solving the linearized form of the governing equations, we obtain a dispersion relation:

$$(\omega^2-\omega_M^2-\omega_A^2)(\omega^2-\omega_M^2)-\omega_C^2\omega^2 = 0 \tag{7}$$

The dispersion relation yields two sets of roots, which correspond to the fast and slow MAC waves.