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Loading kappa-type distributions in particle simulations

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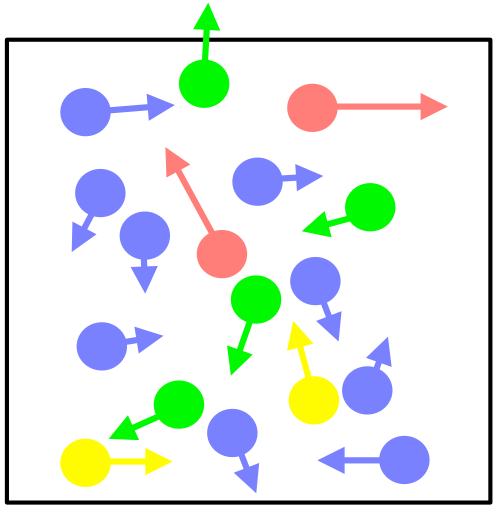
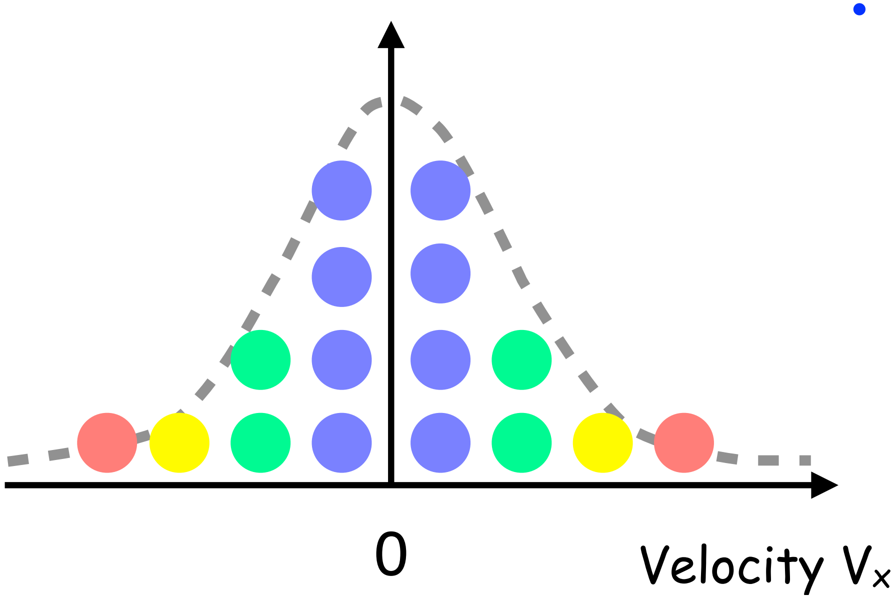
Velocity distributions in particle simulations





- Maxwell-Boltznmann distribution

$$f(\boldsymbol{v})d^3v = N\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} \exp\left(-\frac{mv^2}{2T}\right)d^3v$$

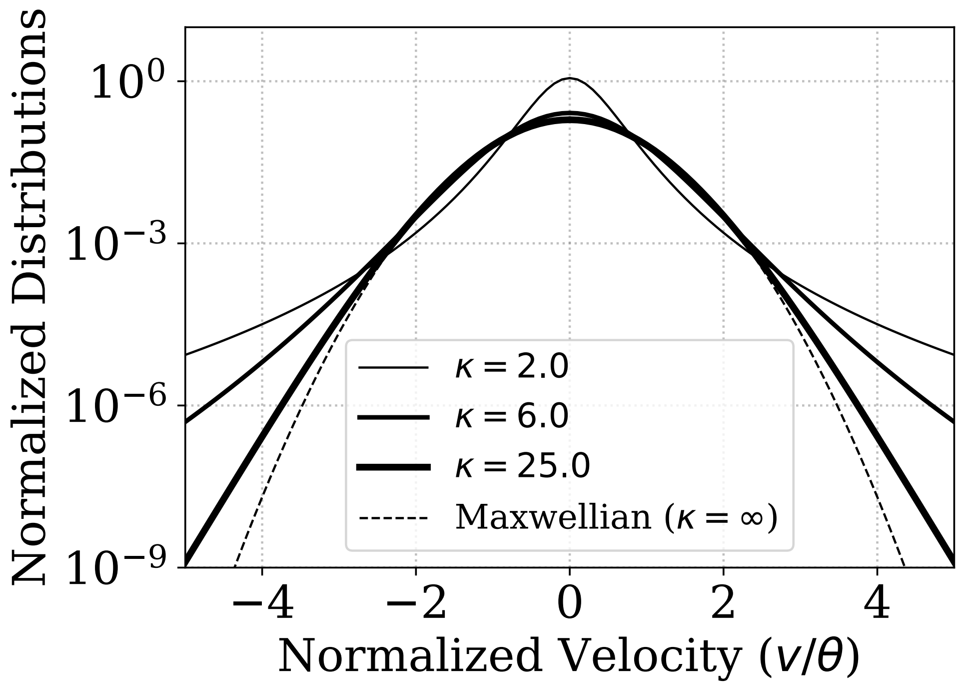
- Box=Muller method (Box & Muller 1958)
  - Two uniform random variates: u_1, $w_1 \subset (0,1]$

$$r_1 = \sqrt{-2\ln u_1}\sin(2\pi w_1)$$
$$r_2 = \sqrt{-2\ln u_1}\cos(2\pi w_1)$$

Kappa distribution

$$f(\boldsymbol{v})d^3v = \frac{N}{(\pi\kappa\theta^2)^{3/2}}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)}\left(1+\frac{v^2}{\kappa\theta^2}\right)^{-(\kappa+1)}d^3v$$

- First introduced in 1968
  (Vasyliunas 1968, Olbert 1968)
- Widely used in a solar-wind plasma with suprathermal ions
- Thermal core + power-law tail
- κ (> 3/2) : power-law index
- $\kappa \to \infty$ Maxwellian



Recipe for a Kappa distribution

- Kappa distribution is equivalent to a multivariate-t distribution (Abdul & Mace 2015)

$$\frac{N}{\nu^{p/2}\pi^{p/2}\sigma^{p}}\frac{\Gamma[(\nu+p)/2]}{\Gamma(\nu/2)}\left(1+\frac{1}{\nu}\frac{\boldsymbol{t}^2}{\sigma^2}\right)^{-\frac{(\nu+p)}{2}}$$

- Step 1
  - Load the 3-D normal distribution
- Step 2
  - Load a chi-squared distribution of ν degrees of freedom (Equivalent to a gamma distribution with k=κ-1/2)
- Step 3
  - Divide 1 by 2

Algorithm 1-2

generate $n_1, n_2, n_3 \sim \mathcal{N}(0,1)$

generate $\chi_\nu^2 \sim \mathrm{Ga}(\kappa - 1/2, 2)$

$r \leftarrow \sqrt{\dfrac{\kappa\theta^2}{\chi_\nu^2}}$

$v_x \leftarrow r n_1$

$v_y \leftarrow r n_2$

$v_z \leftarrow r n_3$

Algorithm A

function Gamma-generator(k, λ)

generate uniform random $U_1, U_2, \cdots, U_{[k]} \in (0,1]$

if k is an integer **then**

$x \leftarrow -\ln\left(\prod_{i=1}^{k} U_i\right)$

elseif k is a half integer **then**

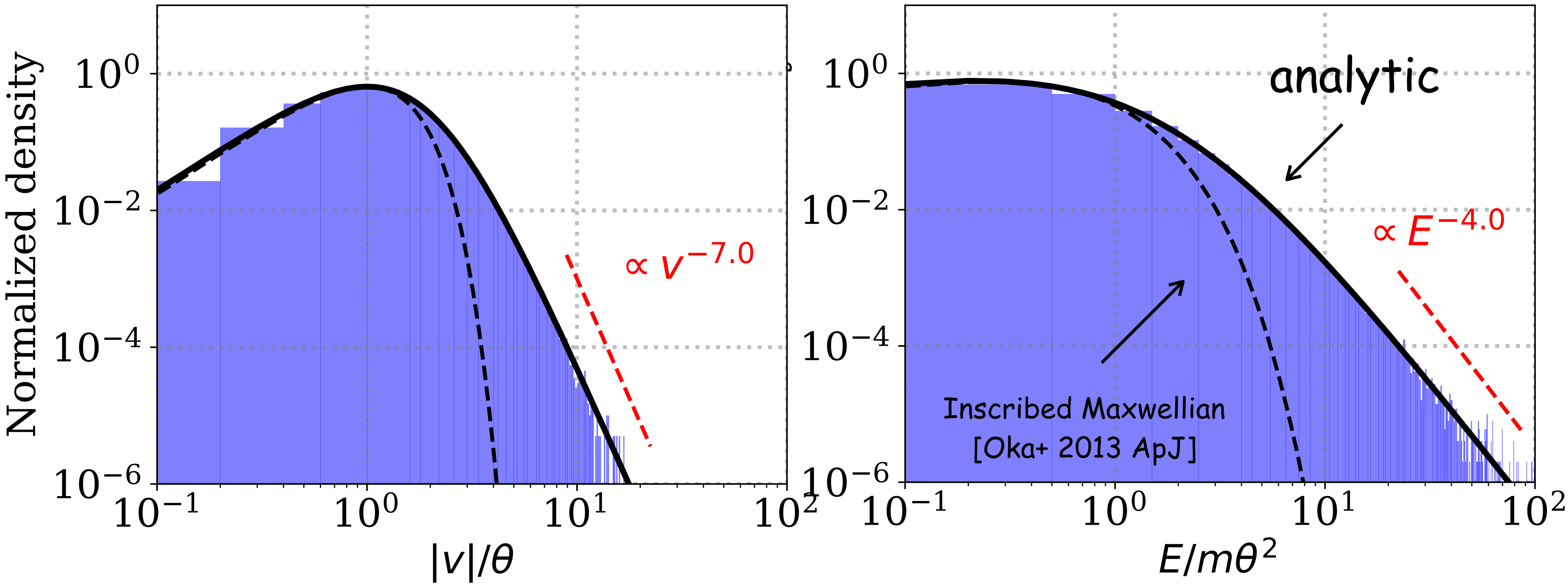
generate $n \sim \mathcal{N}(0,1)$

$x \leftarrow -\ln\left(\prod_{i=1}^{[k]} U_i\right) + \frac{1}{2}n^2$

endif

return λx

Monte-Carlo (PIC) results: κ=3.5

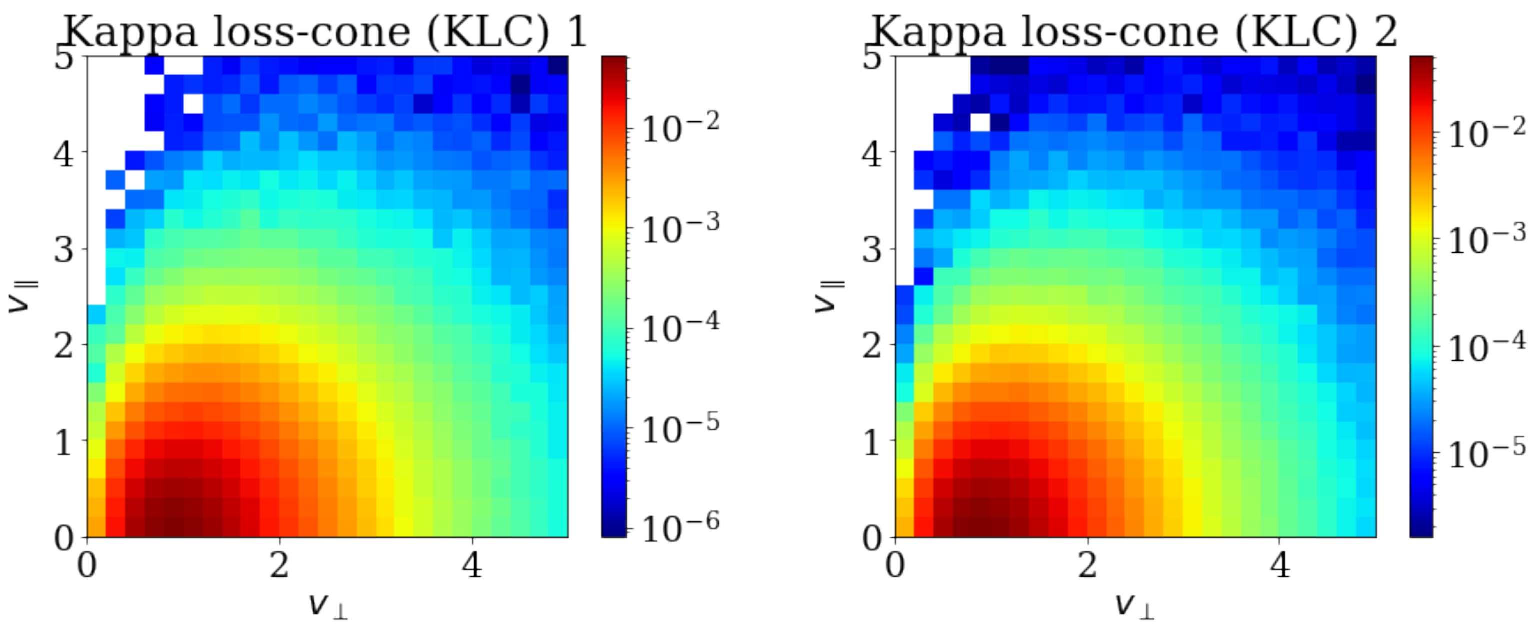


- Power-law tail of ~v**(2κ), ~E**(κ+1/2)
- It extends well beyond the inscribed Maxwellian

Kappa loss-cone (KLC) distribution

- Kappa distribution with an approximated loss-cone (Summers & Thorne 1991)
- Useful for study in the inner magnetosphere

$$f(\boldsymbol{v}) = \frac{N}{\pi^{3/2}\theta_{\parallel}\theta_{\perp}^{2}\kappa^{j+3/2}} \frac{\Gamma(\kappa+j+1)}{\Gamma(j+1)\Gamma(\kappa-1/2)} \boxed{\left(\frac{v_{\perp}}{\theta_{\perp}}\right)^{2j}} \left(1+\frac{{v_{\parallel}}^{2}}{\kappa\theta_{\parallel}^{2}}+\frac{{v_{\perp}}^{2}}{\kappa\theta_{\perp}^{2}}\right)^{-(\kappa+j+1)}$$



Kappa loss-cone (KLC) distribution

- As results of lengthy calculation,
  we have developed two procedures at this point.

$\nu \leftarrow 2\kappa - 1$

$X \leftarrow N(1,0)$

$Y \leftarrow \mathrm{Gamma}(\nu/2, 2)$

$R \leftarrow (1 + X^2/Y)\,\dfrac{\mathrm{Gamma}(j+1,1)}{\mathrm{Gamma}(\kappa,1)}$

$\bar{v}_x \leftarrow \theta_{\parallel}\sqrt{\kappa}\; X/\sqrt{Y}$

$\bar{v}_y \leftarrow \theta_{\perp}\sqrt{\kappa R}\sin(2\pi w_2)$

$\bar{v}_z \leftarrow \theta_{\perp}\sqrt{\kappa R}\cos(2\pi w_2)$

return v_x, v_y, v_z

function $\mathrm{Gamma}(k, \lambda)$

if k is a half integer **then**

  generate uniform random $U_1, U_2, \cdots, U_{k-1/2} \in (0,1], u_1, u_2 \in (0,1]$

  $X \leftarrow -2\lambda \left\{ \ln\left(\prod_{i=1}^{k-1/2} U_i\right) + \cos^2(2\pi u_1)\ln u_2 \right\}$

elseif k is an integer **then**

  generate uniform random $U_1, U_2, \cdots, U_k \in (0,1]$

  $X \leftarrow -2\lambda \ln\left(\prod_{i=1}^{k} U_i\right)$

endif

return X

$X \leftarrow N(1,0)$

$Y \leftarrow \mathrm{Gamma}(\mu/2, 2)$

$R \leftarrow \dfrac{\mathrm{Gamma}(j+1,1)}{\mathrm{Gamma}(\kappa-1/2,1)}$

$\bar{v}_x \leftarrow \theta_{\parallel}\sqrt{\kappa}\; X\sqrt{\dfrac{(1+R)}{Y}}$

$\bar{v}_y \leftarrow \theta_{\perp}\sqrt{\kappa R}\sin(2\pi w_2)$

$\bar{v}_z \leftarrow \theta_{\perp}\sqrt{\kappa R}\cos(2\pi w_2)$

return v_x, v_y, v_z

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  generate uniform random $U_1, U_2, \cdots, U_k \in (0,1]$

  $X \leftarrow -2\lambda \ln\left(\prod_{i=1}^{k} U_i\right)$

endif

return X

Relativistic Kappa distribution

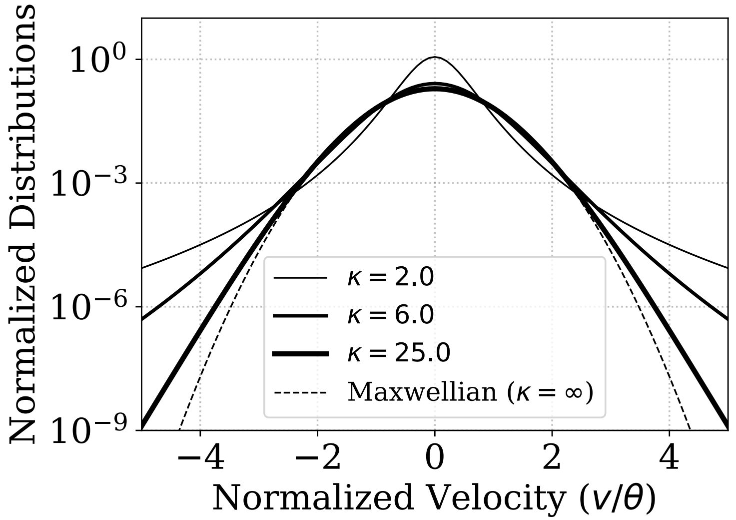
- Kappa distribution

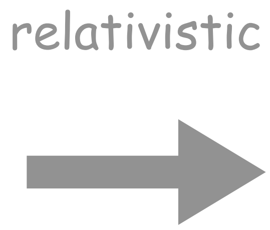
$$f(\boldsymbol{v})d^3v = \frac{N}{(\pi\kappa\theta^2)^{3/2}}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)}\left(1+\frac{v^2}{\kappa\theta^2}\right)^{-(\kappa+1)}d^3v$$

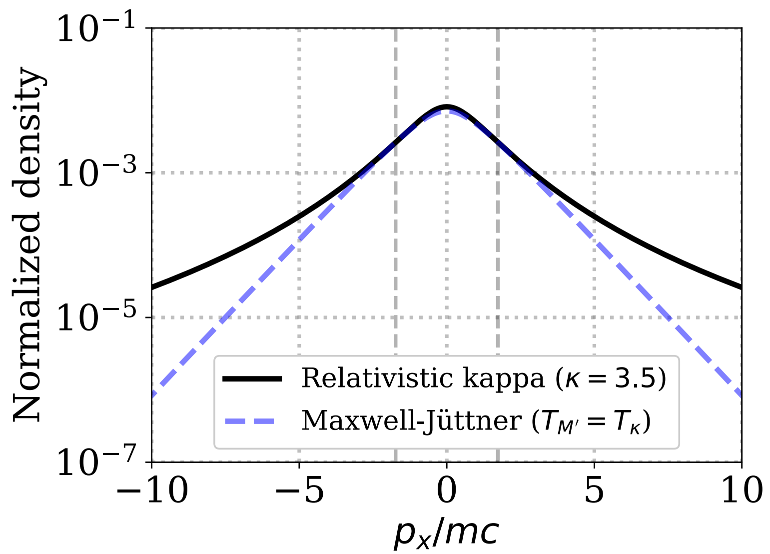
- Relativistic Kappa distribution

$$\boxed{f_{\mathrm{RK}}(\boldsymbol{p})d^3p \equiv A\left(1+\frac{(\gamma-1)mc^2}{\kappa T_\kappa}\right)^{-(\kappa+1)} d^3p}$$

$$\boldsymbol{p} = m\gamma\boldsymbol{v} \qquad \gamma = [1-(v/c)^2]^{-1/2}$$







- Useful for study in high-energy astrophysics & particle physics
- Previous "power-law" models often assume a sudden low-energy cutoff

Some mathematics

- Relativistic Kappa distribution

$$f_{\mathrm{RK}}(p)dp = A(\kappa,t)\left(1+\frac{\gamma-1}{\kappa t}\right)^{-(\kappa+1)} 4\pi p^2\, dp$$

$$x \equiv \frac{\mathcal{E}_{\mathrm{kin}}}{mc^2} = \gamma - 1$$

Normalization factors

$$A(\kappa,T_\kappa) = \frac{N_\kappa \Gamma\left(\kappa+\frac{1}{2}\right)}{(2\pi m\kappa T_\kappa)^{3/2}(\kappa+1)\ \Gamma(\kappa-2)\ {}_2F_1\left(-\frac{3}{2},\frac{5}{2};\kappa+\frac{1}{2};1-\frac{\kappa T_\kappa}{2mc^2}\right)}$$

$$S(\kappa,t) \equiv \frac{\sqrt{2\pi}}{2}\Gamma\left(\kappa-\frac{1}{2}\right) + a\sqrt{\kappa t}\ \Gamma(\kappa-1) + b\ \frac{3\sqrt{2\pi}}{4}(\kappa t)\ \left(\kappa-\frac{3}{2}\right) + 2(\kappa t)^{3/2}\ \Gamma(\kappa-2)$$

- Our expression

$$f_{\mathrm{RK}}(x)dx = \frac{4\pi A(\kappa,t)S(\kappa,t)}{\Gamma(\kappa+1)}(\kappa t)^{3/2}\left(\sum_{i=3}^{6}\pi_i(\kappa,t)\ \mathrm{B}'\left(x;\frac{i}{2},\kappa+1-\frac{i}{2},1,\kappa t\right)\right)R(x)\ dx$$

Probability

$$\sum_{i=3}^{6}\pi_i(\kappa,t) = 1$$

Generalized Beta prime distribution

$$\mathrm{B}'(x;\alpha,\beta,p,q) = \frac{p}{qB(\alpha,\beta)}\left(\frac{x}{q}\right)^{\alpha p-1}\left(1+\left(\frac{x}{q}\right)^p\right)^{-(\alpha+\beta)}$$

Rejection function

$$R(x;a,b) \equiv \frac{(1+x)\sqrt{x+2}}{\sqrt{2}+ax^{1/2}+b\sqrt{2}x+x^{3/2}}$$

The algorithm

Main procedure

$a \leftarrow 0.56, \quad b \leftarrow 0.35, \quad R_0 \leftarrow 0.95$

compute π_3, π_4, π_5 for given κ, t using Eqs. (40)–(42)

repeat

generate $X_1, X_2 \sim U(0,1)$

Probabilistic switch

if $X_1 < \pi_3$ **then** $i \leftarrow 3$

elseif $X_1 < \pi_3 + \pi_4$ **then** $i \leftarrow 4$

elseif $X_1 < \pi_3 + \pi_4 + \pi_5$ **then** $i \leftarrow 5$

else $i \leftarrow 6$

endif

generate $X_3 \sim \mathrm{Ga}(i/2, 1), X_4 \sim \mathrm{Ga}(\kappa + 1 - i/2, 1)$

$x \leftarrow \kappa t \times \dfrac{X_3}{X_4}$

Beta prime distribution

until $X_2 < R_0$ **or** $X_2 < R(x; a, b)$

Rejection

generate $X_5, X_6 \sim U(0,1)$

$p \leftarrow \sqrt{x\,(x+2)}$

$p_x \leftarrow p\,(2X_5 - 1)$

$p_y \leftarrow 2p\sqrt{X_5(1-X_5)}\cos(2\pi X_6)$

$p_z \leftarrow 2p\sqrt{X_5(1-X_5)}\sin(2\pi X_6)$

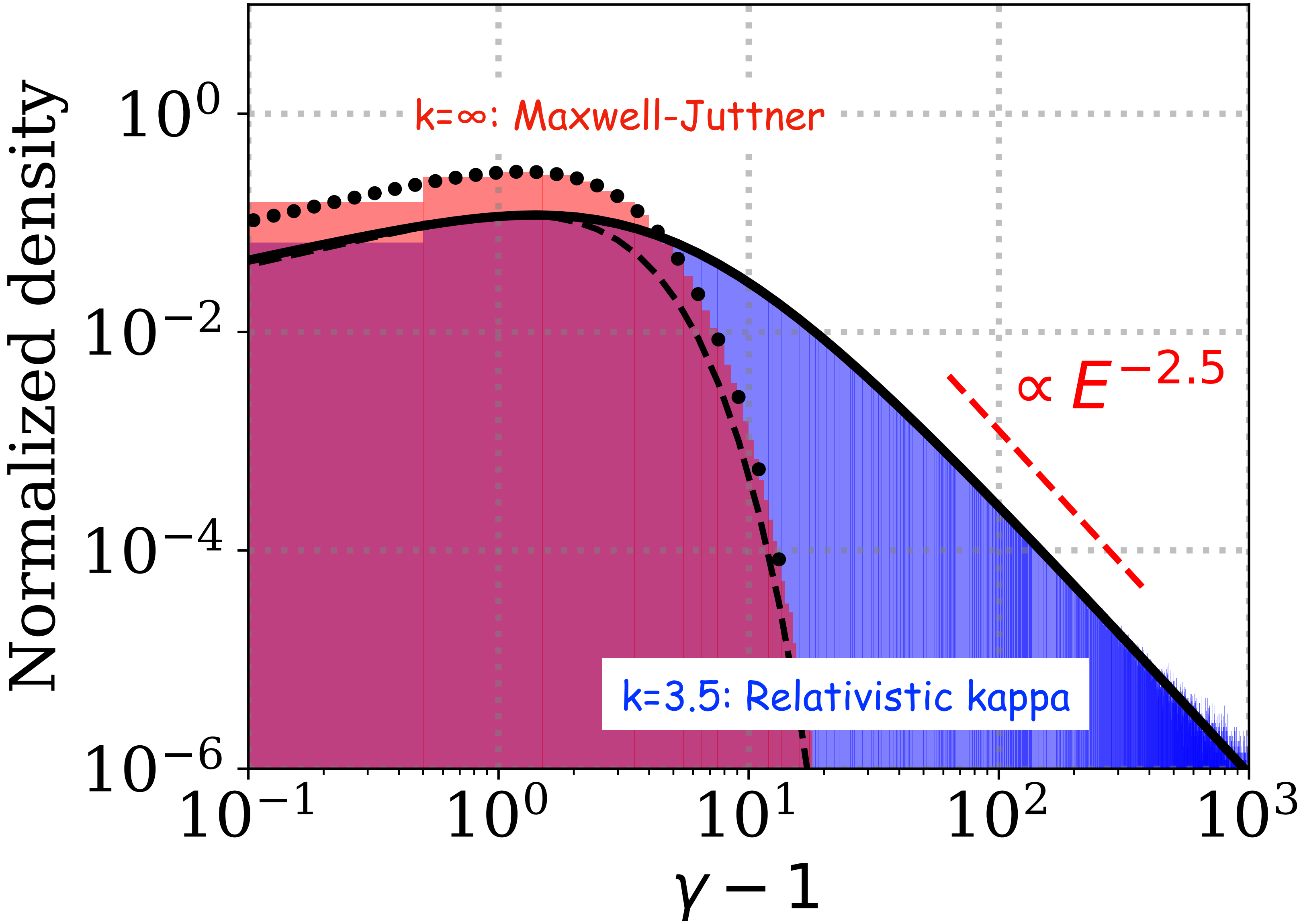
- Beta prime distribution can be generated from 2 gamma distributions

$$X_{\mathrm{B}'(\alpha,\beta)} = \frac{X_{\Gamma(\alpha,\delta)}}{X_{\Gamma(\beta,\delta)}} = \frac{X_{\Gamma(\alpha,1)}}{X_{\Gamma(\beta,1)}}$$

- No need for the normalization factors
- Gamma variates can be easily generated

- Rejection function will be discussed in the last section

Numerical test

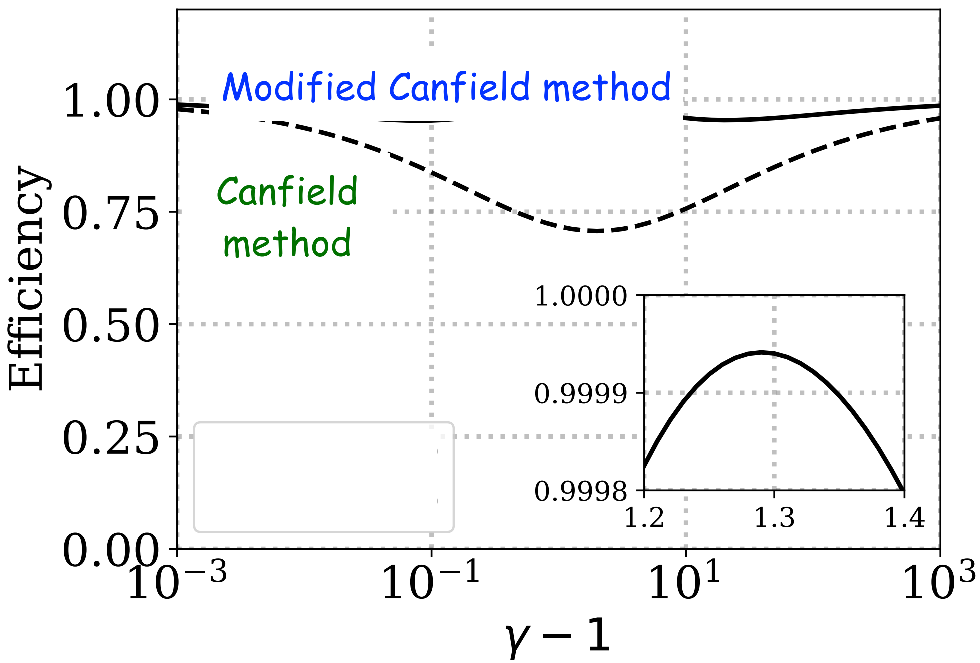


Zenitani & Nakano 2022

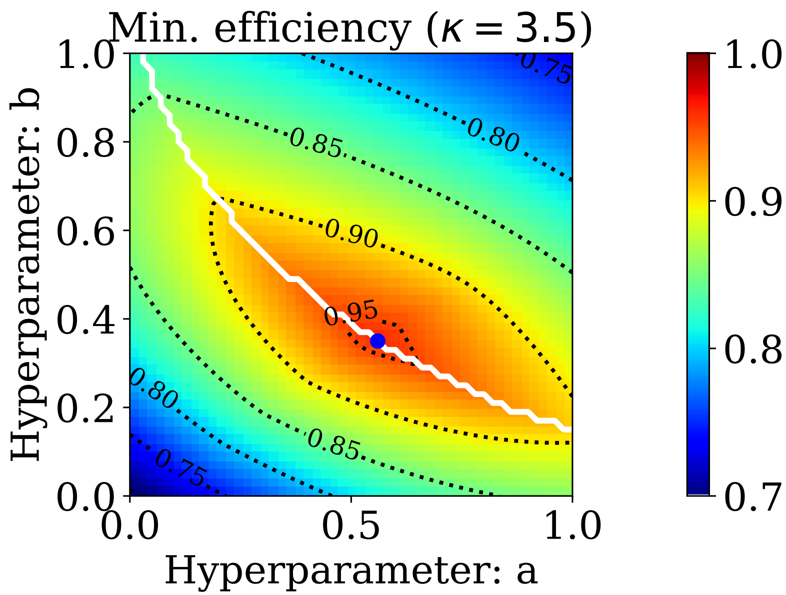
Acceptance efficiency (1/2)

- In the relativistic cases, we need a rejection method to adjust the distribution.

$$R(x;a,b) \equiv \frac{(1+x)\sqrt{x+2}}{\sqrt{2}+ax^{1/2}+b\sqrt{2}x+x^{3/2}}$$



- We have added two hyper-parameters to a rejection function by Canfield+ (1987)
- Best hyperparameters are found by a grid-search

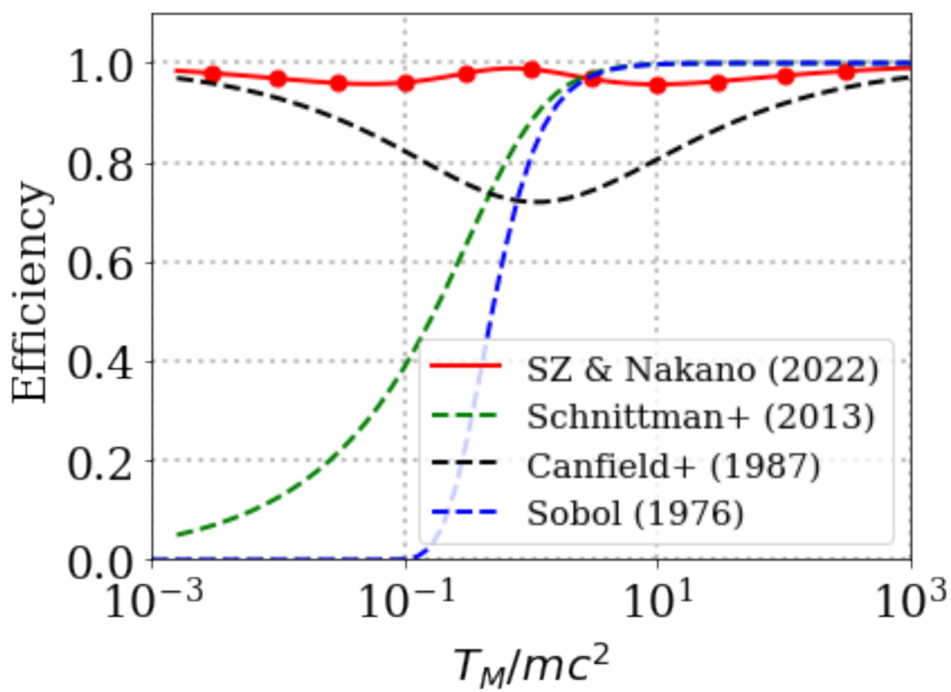


Zenitani & Nakano 2022

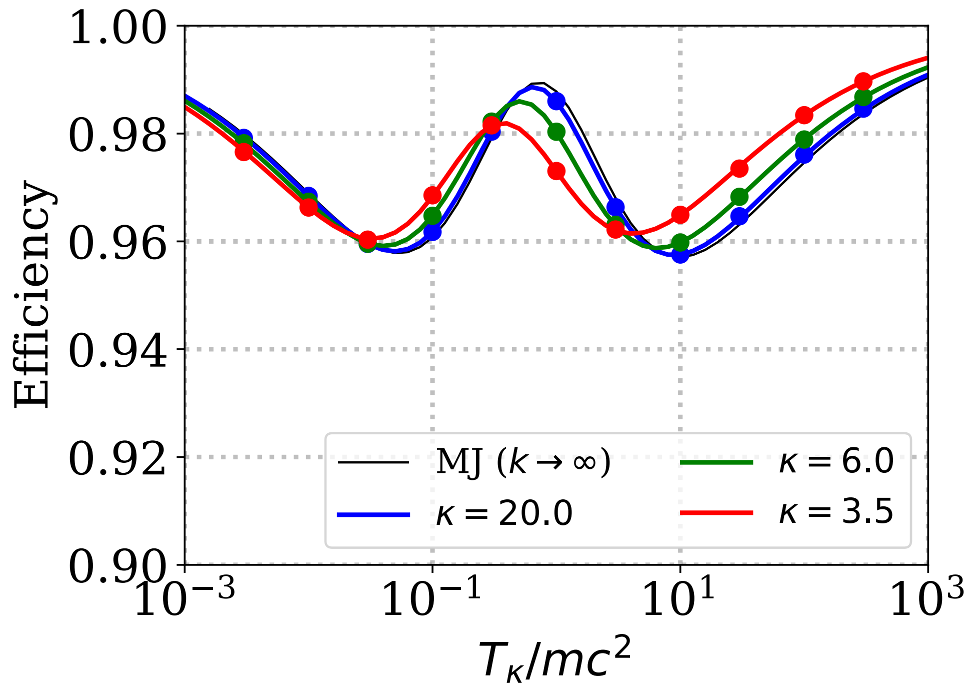
Acceptance efficiency (2/2)

- The new function improves efficiency for relativistic kappa distributions as well as Maxwell-Jüttner distributions (relativistic Maxwell distributions)
- Drastic improvement: 70% --> 95%

- Maxwell=Jüttner distributions



- Relativistic Kappa distributions



Summary

- 1. Kappa distribution <-- Multivariate-t distribution
- 2. Kappa loss-cone distribution <-- Lengthy calculation
- 3. Relativistic kappa distribution <-- Beta prime distributions
- 4. Efficient rejection function for relativistic distributions

- References:
  - Zenitani & Nakano, *Phys. Plasmas* **29**, 113904 (2022) 3 & 4
  - Zenitani & Nakano, *in prep.* 2 & loss-cone distributions