Natural measures of asymptotically autonomous systems

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Parameter-dependent multistable climate system "without climate change"

Autonomous ODE

$$\dot{x} = f(\lambda, x) = f_\lambda(x)$$

generating semiflow $(\varphi_\lambda^t)_{t\geq 0}$ on a compact subset of Euclidean space, with

- mutually disjoint attractors ("stable states of the climate system")

$$A_{1,\lambda}, A_{2,\lambda}, \ldots, A_{n(\lambda),\lambda}$$

  for all but at-most-finitely-many λ-values, where
- $A_{i,\lambda}$ *not* a singleton, ...

→ the quantitatively precise climate state is always changing!
- ... but is instead the *support of a natural probability distribution* $\mu_{i,\lambda}$.

"Natural": $\exists$ neighbourhood U of $A_{i,\lambda}$ s.t.

$$\forall\, g \in C_b(U, \mathbb{R}), h \in L^1(U, \mathbb{R}),$$
$$\frac{1}{T}\int_U \int_0^T g(\varphi_\lambda^t x)\, dt\, h(x)\, dx \;\to\; \mathbb{E}_{\mu_{i,\lambda}}[g] \int_U h(x) dx \text{ as } T \to \infty.$$

Heuristic interpretation: For each i, $\mu_{i,\lambda}$ is the probability distribution for what the current quantitative climate state $x(t_{\text{now}})$ is, *conditional on* the knowledge that for $t \approx -\infty$, the state $x(t)$ was near $A_{i,\lambda}$.

Now introduce "climate change"

Nonautonomous ODE

$$\dot{x} = f(\Lambda(t), x), \qquad \Lambda(t) \to \lambda_\pm \text{ as } t \to \pm\infty$$

generating nonautonomous semiflow $(\varphi^{s\to t})_{-\infty < s \leq t < \infty}$.

$\Lambda(\cdot)$ represents "real-time (as opposed to quasistatic) parameter drift" $\Longrightarrow$ **rate-induced** phenomena.

Definition. Given $i \in \{1, \ldots, n(\lambda_-)\}$, a **natural probability distribution rooted at** A_{i,λ_-} is a time-dependent probability measure $(\mu_i^t)_{t\in\mathbb{R}}$ such that:

$\exists$ neighbourhood U of A_{i,λ_-} s.t.

$$\forall\, g \in C_b(U, \mathbb{R}), h \in L^1(U, \mathbb{R}), t \in \mathbb{R},$$
$$\frac{1}{T}\int_U \int_{t-T}^t g(\varphi^{s\to t} x)\, ds\, h(x)\, dx \;\to\; \mathbb{E}_{\mu_i^t}[g] \int_U h(x) dx \text{ as } T \to \infty.$$

Heuristic interpretation: If $(\mu_i^t)_{t\in\mathbb{R}}$ exists then $\mu_i^{t_{\text{now}}}$ is the probability distribution for what the current quantitative climate state $x(t_{\text{now}})$ is, *conditional on* the knowledge that for $t \approx -\infty$, the state $x(t)$ was near A_{i,λ_-}.

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Tipping probability

Tipping. Given A_{i,λ_-} for some $i \in \{1, \ldots, n(\lambda_-)\}$ and A_{j,λ_+} for some $j \in \{1, \ldots, n(\lambda_+)\}$, with A_{i,λ_-} and A_{j,λ_+} *not* connected by a continuous branch of attractors $(A_{i(\Lambda(t)),\Lambda(t)})_{t\in\mathbb{R}}$: we say that the climate system *tips from* A_{i,λ_-} *to* A_{j,λ_+} if

- in the distant past, the quantitative climate state $x(t)$ was near A_{i,λ_-};
- in the distant future, the quantitative climate state $x(t)$ will be near A_{j,λ_+}.

For singleton models of stable climate states, rate-induced tipping has been studied in [2].

For non-singleton attractor models of stable climate states, we now ask about **probability of tipping** [3].

Theorem [1]. Fix i and assume $(\mu_i^t)_{t\in\mathbb{R}}$ exists. Assume "reasonable assumptions": the attractors A_{j,λ_+} of $(\varphi_{\lambda_+}^t)_{t\geq 0}$ are all "robust", and for all $\varepsilon > 0$ $\exists$ nbhd O_ε of the complement of the union of the basins of the attractors A_{j,λ_+} such that $\limsup_{t\to\infty} \mu_i^t(O_\varepsilon) < \varepsilon$. Then:

$\exists\, p_{i,1}, \ldots, p_{i,n(\lambda_+)} \in [0,1]$ with $\sum_{j=1}^{n(\lambda_+)} p_{i,j} = 1$ s.t. for each j, for any nbhd U of A_{j,λ_+} contained in the basin of A_{j,λ_+}, we have

$$\mu_i^t(U) \to p_{i,j} \text{ as } t \to \infty.$$

We call $p_{i,j}$ the **probability of tipping from** A_{i,λ_-} **to** A_{j,λ_+}.

Existence of the natural probability distribution $(\mu_i^t)_{t\in\mathbb{R}}$

Assume that $\Lambda(\cdot)$ is monotonically increasing. Heuristically, if "$\Lambda(t) \to \lambda_-$ sufficiently fast as $t \to -\infty$" then $(\mu_i^t)_{t\in\mathbb{R}}$ exists.

Theorem [4]. Fix i. Assume that the attractor A_{i,λ_-} of $(\varphi_{\lambda_-}^t)_{t\geq 0}$ and its natural distribution μ_{i,λ_-} are "robust" with "linear-response-like behaviour". Let

$$r_0 = \sup\left(\left\{\frac{(x-y)\cdot(f(\lambda_-,x) - f(\lambda_-,y))}{|x-y|^2} : x, y \in A_{i,\lambda_-},\, x \neq y\right\} \cup \left\{\text{max eigenvalue of } \tfrac{1}{2}\left(\tfrac{\partial f}{\partial x}(\lambda_-, x) + \tfrac{\partial f}{\partial x}(\lambda_-, x)^{\mathrm{T}}\right) : x \in A_{i,\lambda_-}\right\}\right).$$

If there exists $r > r_0$ such that

$$\int_{-\infty}^{\text{arbitrary finite number}} e^{-rt}\left(d_{\text{1-Wass}}(\mu_{i,\Lambda(t)}, \mu_{i,\lambda_-}) \;+\; d_{\text{Haus}}(A_{i,\Lambda(t)}, A_{i,\lambda_-}) \;+\; d_{\sup}(f_{\Lambda(t)}, f_{\lambda_-})\right) dt < \infty$$

then there exists a natural probability distribution rooted at A_{i,λ_-}.

Open question

The "linear-response-like behaviour" is specifically: there exist $\delta > 0$ and $C \geq 1$ s.t. for each $\lambda \in (\lambda_-, \lambda_- + \delta)$ one can find $\tilde{\delta}(\lambda) > 0$ s.t. for all $\lambda' \in (\lambda, \lambda + \tilde{\delta}(\lambda))$,

$$d_{\text{1-Wass}}(\mu_{i,\lambda}, \mu_{i,\lambda'}) \leq C(d_{\text{1-Wass}}(\mu_{i,\lambda'}, \mu_{i,\lambda_-}) \;-\; d_{\text{1-Wass}}(\mu_{i,\lambda}, \mu_{i,\lambda_-}))$$
$$d_{\text{Haus}}(A_{i,\lambda}, A_{i,\lambda'}) \leq C(d_{\text{Haus}}(A_{i,\lambda'}, A_{i,\lambda_-}) \;-\; d_{\text{Haus}}(A_{i,\lambda}, A_{i,\lambda_-})).$$

Question: Does this follow from typical linear response assumptions? And/Or is this reasonable to expect if A_{i,λ_-} is a "sufficiently nice" (e.g. uniformly hyperbolic) attractor of $(\varphi_{\lambda_-}^t)_{t\geq 0}$ (assuming f is C^∞ in (λ, x))? Alternatively, can this condition be dropped?

References

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