On the statement and numerical solution of the thermal problem within inversion methods for the study of lithospheric structure.

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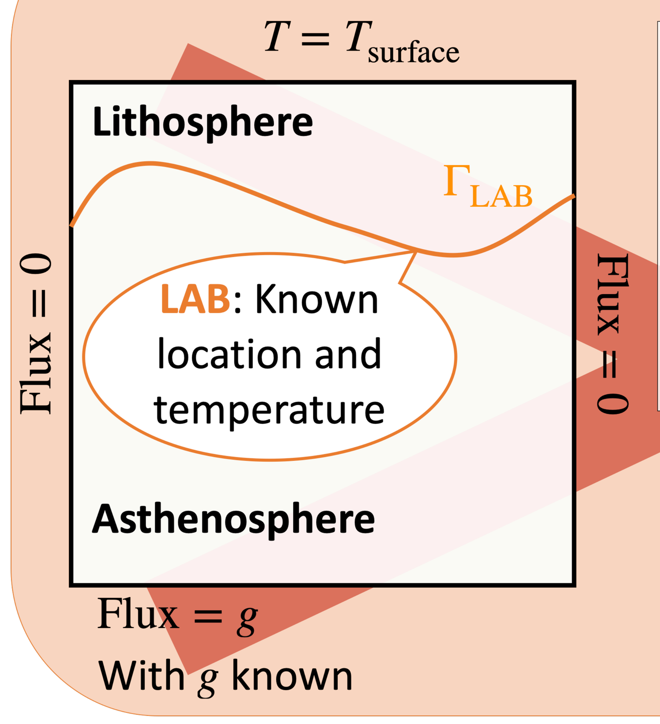
1. PROBLEM STATEMENT

- Ortega et al. [1] proposed a reduced basis procedure to compute the Stokes problem within a multi-observable inversion. In that case, density and viscosity properties were based on a simplistic temperature field. **Our goal is to extend its capabilities by computing the thermal field as the solution of an energy balance equation.**
- A fixed computational mesh is used to facilitate the interaction with the inverse solver.
- Input: LAB location ($T_{LAB} = 1500$ K); Output: thermal field.

OVER-CONSTRAINED FORWARD PROBLEM

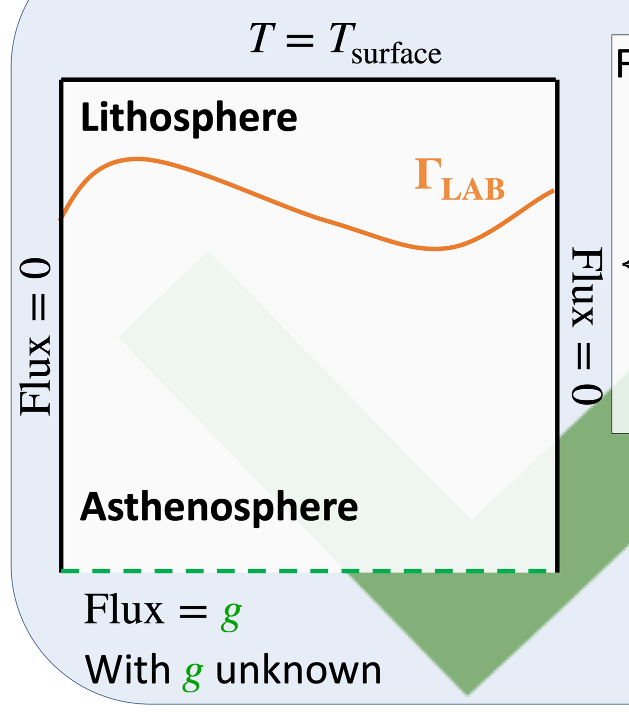


Find T such that,

$$\begin{cases} -\nabla(\kappa \nabla T) + \rho c \mathbf{u} \cdot \nabla T = s, & \text{in } \Omega, \\ T = T_{surface}, & \text{on } \Gamma_{surface}, \\ -\kappa \nabla T \cdot \mathbf{n} = 0, & \text{on } \Gamma_{walls}, \\ -\kappa \nabla T \cdot \mathbf{n} = g, & \text{on } \Gamma_{bottom}, \\ T = T_{LAB}, & \text{on } \Gamma_{LAB}. \end{cases}$$

- The thermal problem has a unique solution given its boundary conditions. The LAB immersed condition, over-determines the problem.
- Well posedness is recovered by relaxing g given this is the boundary condition known with the highest uncertainty.

WELL-POSED RELAXED FORWARD PROBLEM



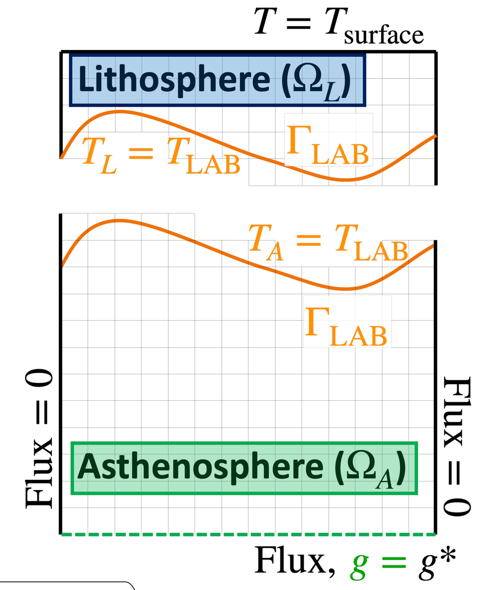
Find T and g such that,

$$\begin{cases} -\nabla(\kappa \nabla T) + \rho c \mathbf{u} \cdot \nabla T = s, & \text{in } \Omega, \\ T = T_{surface}, & \text{on } \Gamma_{surface}, \\ -\kappa \nabla T \cdot \mathbf{n} = 0, & \text{on } \Gamma_{walls}, \\ -\kappa \nabla T \cdot \mathbf{n} = g, & \text{on } \Gamma_{bottom}, \leftarrow \text{RELAX} \\ T = T_{LAB}, & \text{on } \Gamma_{LAB}. \leftarrow \text{ENFORCE} \end{cases}$$

2. PROPOSED METHOD

The problem is divided into two sub-domains, Lithosphere and Asthenosphere. In all cases $\mathbf{u}_L$ and $\mathbf{u}_A$ are considered known.

- In the Lithosphere the problem is well-posed and all boundary conditions are known: T_L is obtained.
- The Asthenosphere problem is well-posed but g is unknown,
  - Lithospheric flux onto the LAB is assessed with T_L results,
  - Asthenopshere flux to the LAB is expressed in terms of g,
  - fluxes differences are minimized to determine g^*,
  - then $T_A(g^*)$ is obtained.



THE RELAXED TWO-STEP SOLUTION

Enforced using Nitsche Method for unfitted meshes

1 Lithosphere problem

Inputs: BC. Find T_L such that,

$$\begin{cases} -\nabla(\kappa_L \nabla T_L) + \rho c\, \mathbf{u}_L \cdot \nabla T_L = s_l, & \text{in } \Omega_L, \\ T_L = T_{surface}, & \text{on } \Gamma_{surface}, \\ -\kappa \nabla T_L \cdot \mathbf{n} = 0, & \text{on } \Gamma_{walls}, \\ T_L = T_{LAB}, & \text{on } \Gamma_{LAB}. \end{cases}$$

($\mathbf{u}_L \approx 0$)

2 Asthenosphere problem

Inputs: BC + T_L. Find g^* such that,

2.1 $$g^* = \arg\min_g ||\kappa_L \nabla T_L \cdot \mathbf{n}_L + \kappa_A \nabla T_A(g) \cdot \mathbf{n}_A||^2$$

and $T_A(g^*)$ such that,

2.2 $$\begin{cases} -\nabla(\kappa_A \nabla T_A) + \rho c\, \mathbf{u}_A \cdot \nabla T_A = s_A, & \text{in } \Omega_A, \\ T_A = T_{LAB}, & \text{on } \Gamma_{LAB}, \\ -\kappa_A \nabla T_A \cdot \mathbf{n} = 0, & \text{on } \Gamma_{walls}, \\ -\kappa_A \nabla T_A \cdot \mathbf{n} = g^*, & \text{on } \Gamma_{bottom}. \end{cases}$$

To avoid non-physical results we modify the minimization problem: the original unrestricted Least-Squares (ULS), and a restricted case (RLS). RLS only allows results associated with $\nabla T \approx 0.35 - 0.5$ K/km as reported by Afonso et al. [2].

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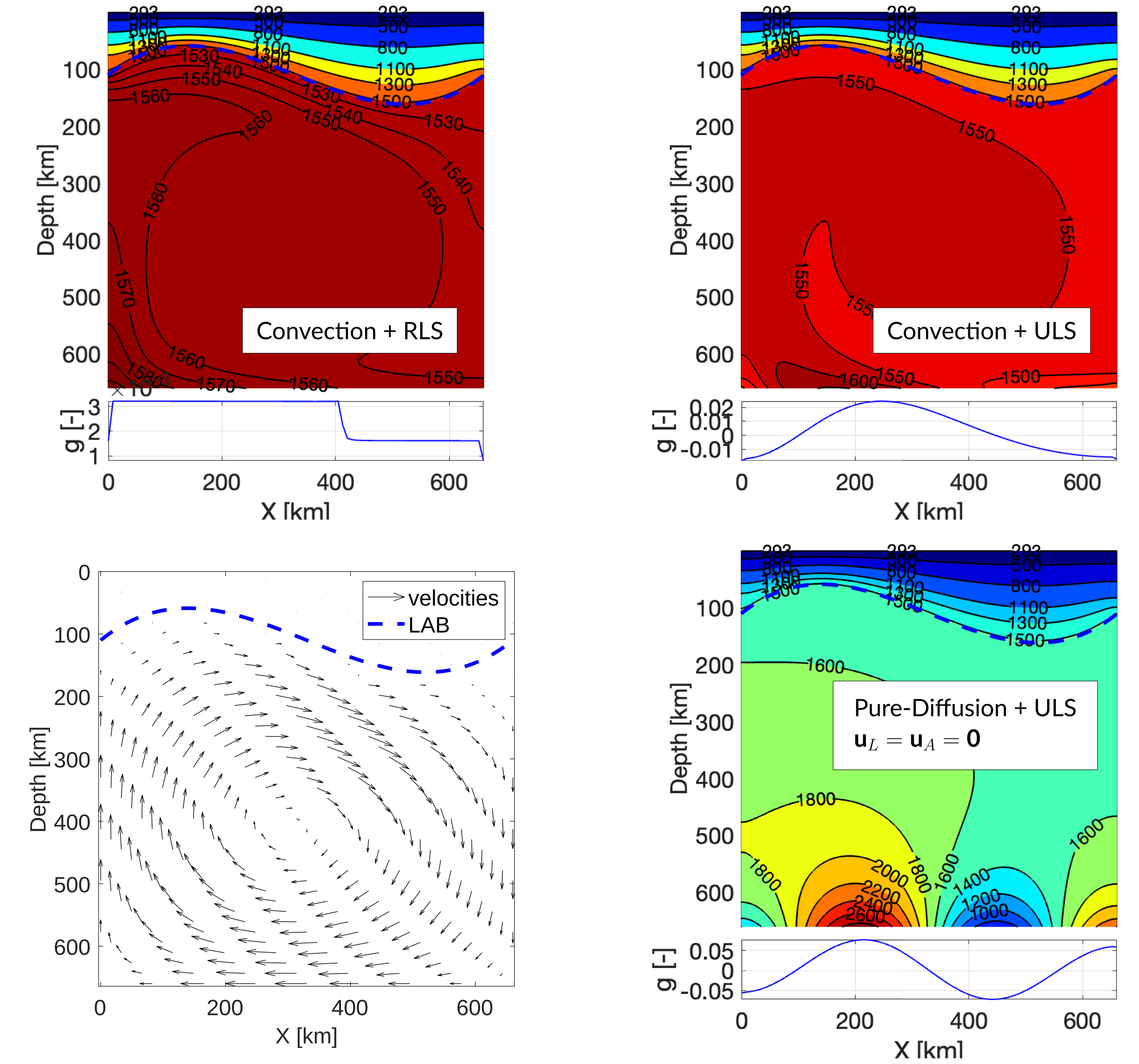
3. RESULTS & COMMENTS



- Pure-diffusion results are physically odd.
- Unrestricted Least-Squares present non-physical results, i.e. $g < 0$, while restricted has similar temperature results respecting physics.

Summary:

- We propose a problem statement and numerical procedure to solve the thermal problem with internal conditions.
- The scheme can be straight forward extended to 3-D domains.
- The proposed method has a computational cost comparable to the original: $\text{cost}(\Omega_L) + \text{cost}(\Omega_A) + \text{cost}(g^*) \approx \text{cost}(\Omega)$.

References

[1] O. Ortega-Gelabert, S. Zlotnik, J. C. Afonso, and P. Díez, "Fast stokes flow simulations for geophysical-geodynamic inverse problems and sensitivity analyses based on reduced order modeling," *Journal of Geophysical Research: Solid Earth*, vol. 125, no. 3, 2020.

[2] J. C. Afonso, M. Fernàndez, G. Ranalli, W. Griffin, and J. Connolly, "Integrated geophysical-petrological modeling of the lithosphere and sublithospheric upper mantle: Methodology and applications," *Geochemistry, geophysics, geosystems*, vol. 9, no. 5, 2008.