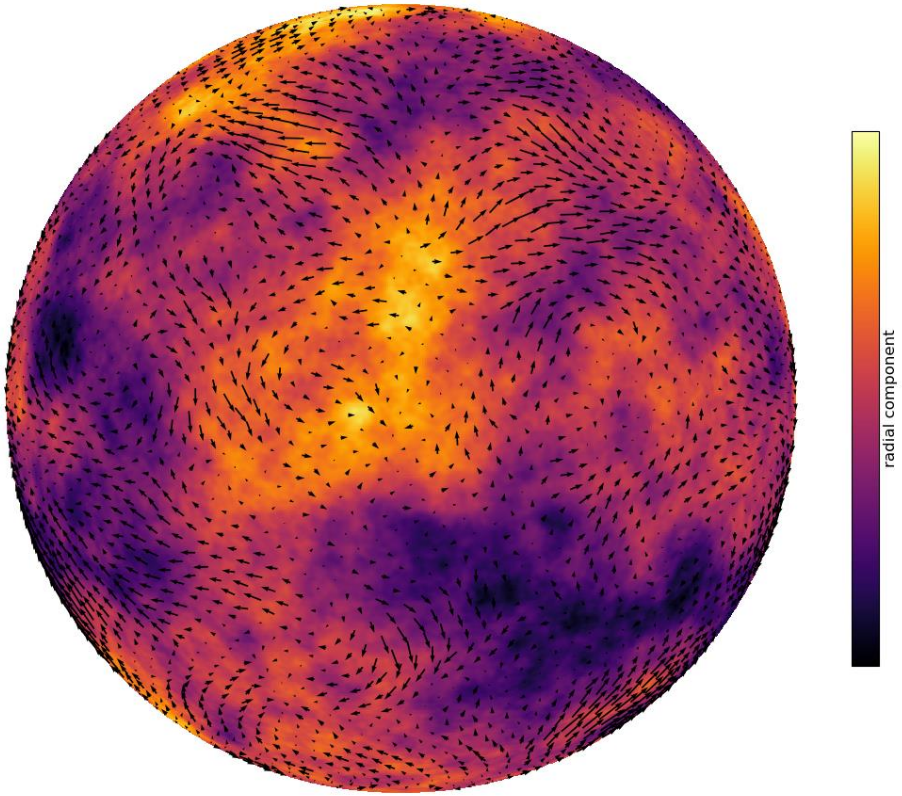
Modeling random isotropic vector fields on the sphere

Paul Rebischung, Kevin Gobron



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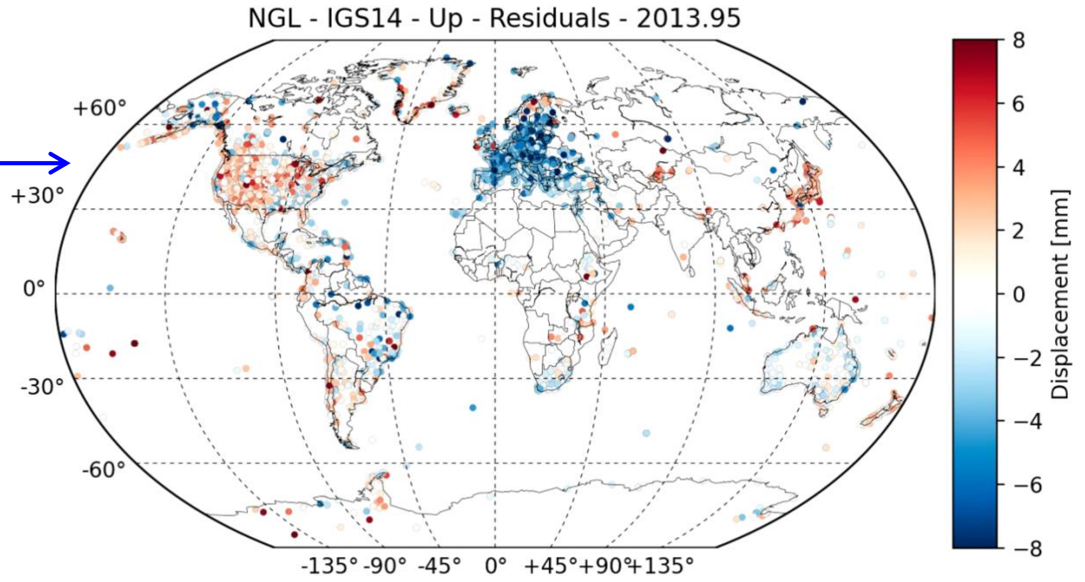
Motivation

- The stochastic variations (a.k.a. noise) in GNSS station position time series are temporally & spatially correlated.
- Temporal correlations are well studied and routinely accounted for in GNSS time series analysis.
  – A 'white + flicker' noise model is usually appropriate and employed.
- Spatial correlations as less well characterized and often ignored, while their consideration could improve:
  – offset detection (Gazeaux et al. 2015),
  – velocity estimation (Benoist et al. 2020),
  – spatial filtering,
  – error budgets for GNSS velocity fields, plate rotation poles, strain maps…



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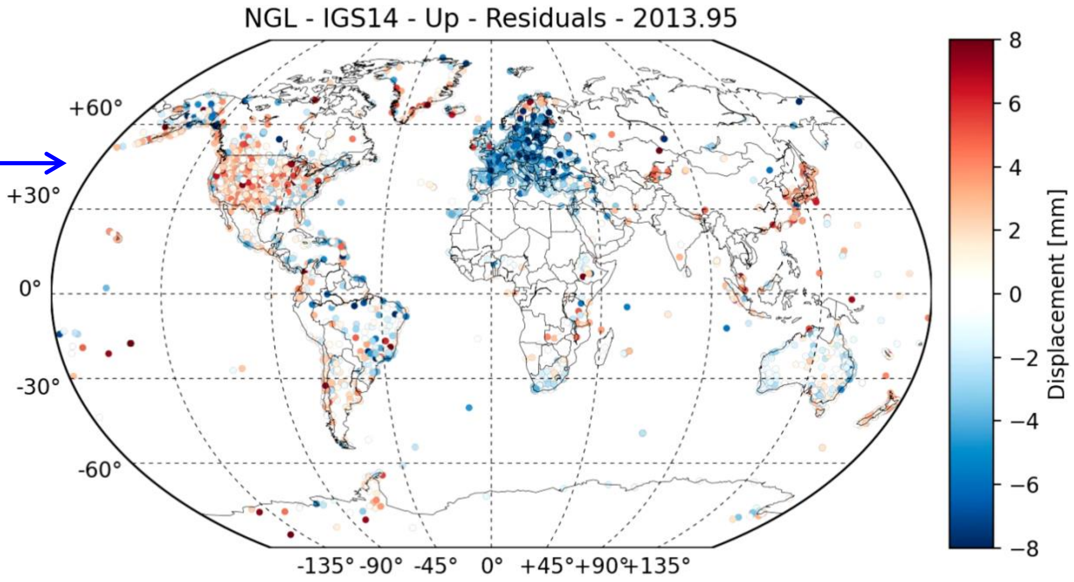


Objective of this work:

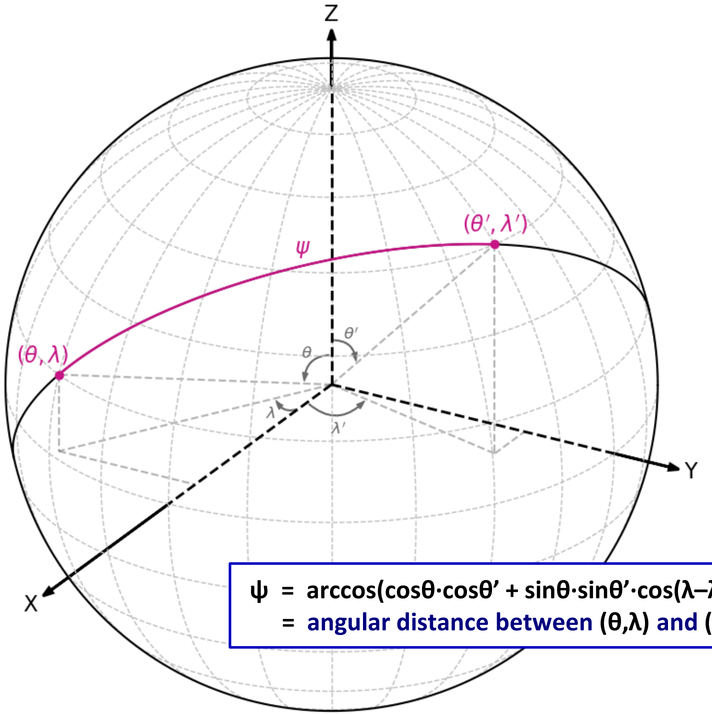
Establish a mathematical framework for the modeling of random vector fields on the sphere.

(such as the field of 3D stochastic GNSS station displacements)

To start with, only random isotropic vector fields are considered.

Random isotropic scalar fields on the sphere

- **The theory of random scalar fields on the sphere is well established.**
  - See for instance the extensive textbook by Marinucci & Peccati (2011).
  - Many recent developments have been motivated by the study of the Cosmological Microwave Background.



ψ = arccos(cosθ·cosθ' + sinθ·sinθ'·cos(λ–λ'))
= angular distance between (θ,λ) and (θ',λ')

- **A random scalar field on the sphere f(θ,λ) is said to be (weakly) isotropic if, for any rotation R:**
  - E[f(R(θ,λ))] = E[f(θ,λ)]
    (Its expected value is rotationally invariant.)
  - cov[f(R(θ,λ)), f(R(θ',λ'))] = cov[f(θ,λ), f(θ',λ')]
    (Its covariance is rotationally invariant.)

- **Isotropy implies:**
  - E[f(θ,λ)] = μ (The expected value is constant over the sphere.)
  - cov[f(θ,λ), f(θ',λ')] = c(ψ) (The covariance only depends on the distance between two points.)
  - Note that c(ψ) needs to satisfy certain conditions to be admissible as a covariance function.

Random isotropic scalar fields on the sphere

- **A random isotropic scalar field on the sphere has the following spectral decomposition:**

$$f(\theta,\lambda) = \mu + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{n,m} Y_{n,m}(\theta,\lambda) \quad \text{with} \quad f_{n,m} = \iint \left(f(\theta,\lambda) - \mu\right) \cdot \overline{Y}_{n,m}(\theta,\lambda) \sin\theta \, d\theta \, d\lambda$$

  - $Y_{n,m}$ = spherical harmonic of degree n, order m (complex, orthonormalized, with Condon-Shortley phase factor)
  - $f_{n,m}$ = random variable (spherical harmonic coefficient of the random field)

- **The $f_{n,m}$ coefficients satisfy:**
  - $E[f_{n,m}] = 0$
  - $f_{n,-m} = (-1)^m \overline{f}_{n,m}$ (since $f(\theta,\lambda)$ is real and $Y_{n,-m} = (-1)^m \overline{Y}_{n,m}$)
  - $\text{cov}[f_{n,m}, f_{n',m'}] = \delta_{n,n'} \delta_{m,m'} C_n$ (They are pairwise uncorrelated, and their variance depends only on the degree n.)
  - If $f(\theta,\lambda)$ is gaussian, the $f_{n,m}$ coefficients of non-negative orders are not only uncorrelated, but also independent.

- **The sequence $(C_n)_{0 \leq n \leq \infty}$ is called the angular power spectrum of the field.**
  - It describes how the variance of the field is distributed across spherical harmonic degrees, i.e., spatial wavelengths.
  - Indeed: $\text{var}\left[f(\theta,\lambda)\right] = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} C_n$

Random isotropic scalar fields on the sphere

- The covariance function $c(\psi)$ and the angular power spectrum (C_n) of a random isotropic scalar field on the sphere are linked through the Legendre transform:

$$\mathrm{cov}\left[f(\theta,\lambda), f(\theta',\lambda')\right] = c(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} C_n \cdot P_n(\cos\psi) \qquad C_n = 2\pi \int_0^{\pi} c(\psi) P_n(\cos\psi) \sin\psi \, d\psi$$

Legendre polynomial of degree n

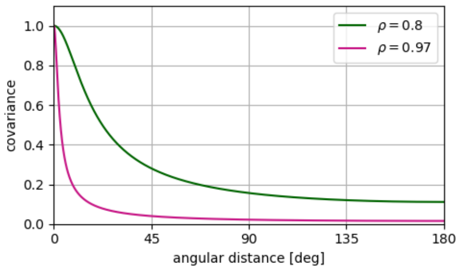
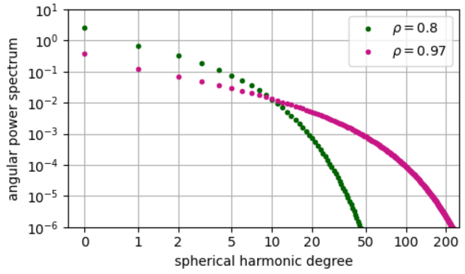
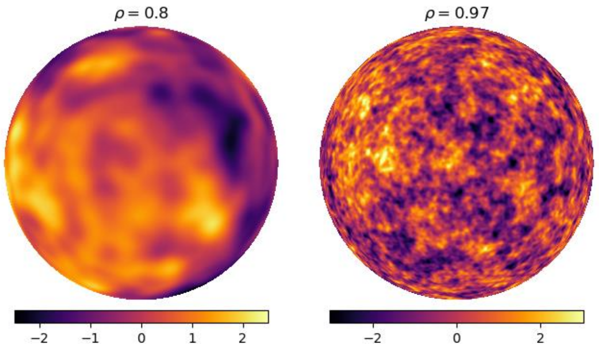
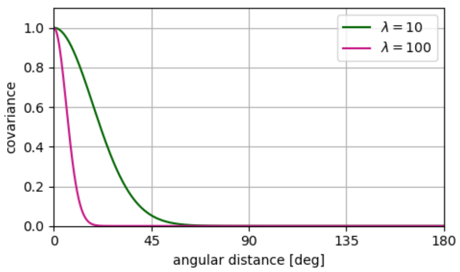
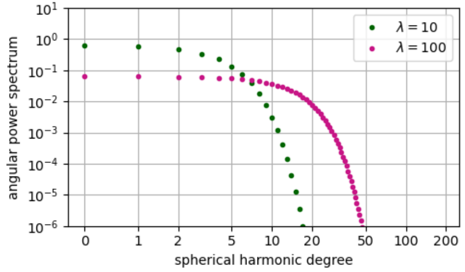
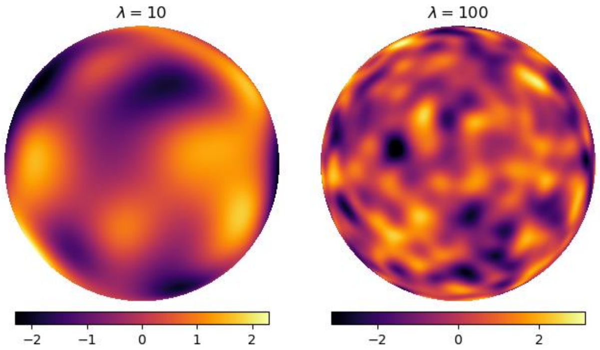
- This result provides the link between the spatial and spectral representations of a random isotropic scalar field on the sphere.

- It is analogous to the Wiener-Khinchin theorem for a stationary random process (of time), which states that its autocovariance and power spectral density are linked through the Fourier transform.

- It also provides a necessary and sufficient condition for $c(\psi)$ to be admissible as a covariance function: the coefficients (C_n) of its Legendre transform need to be positive.

Random isotropic scalar fields on the sphere

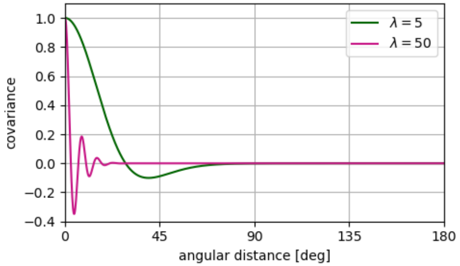
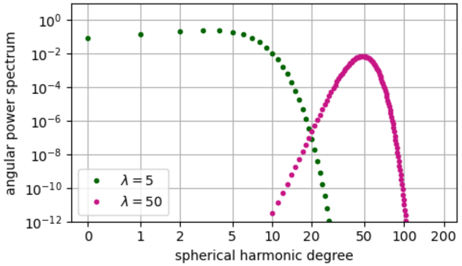
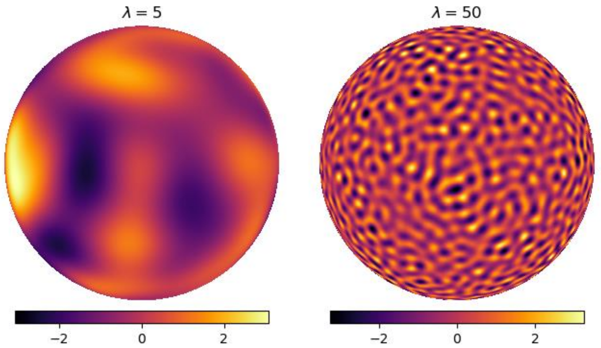
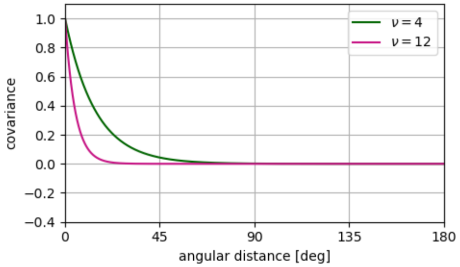
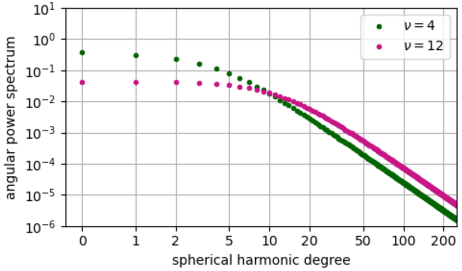
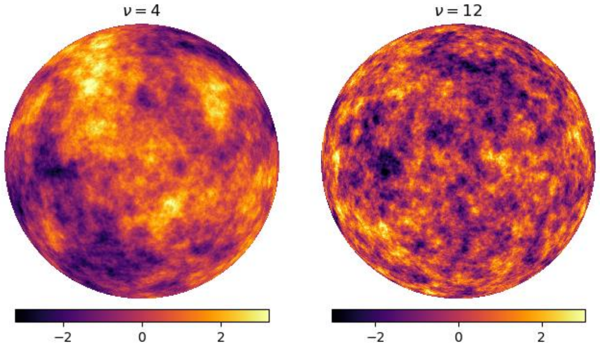
- **Some analytical models:**

Covariance function	Angular power spectrum	Samples
$c(\psi) = (1-\rho)/\sqrt{1-2\rho\cos\psi+\rho^2}$ 	$C_n = 4\pi(1-\rho)\rho^n/(2n+1)$ 	
$c(\psi) = \exp(\lambda(\cos\psi - 1))$ 	$C_n = 2\pi\sqrt{2\pi/\lambda}\,\exp(-\lambda)\,I_{n+1/2}(\lambda)$ 	

I = modified Bessel function of the first kind

Random isotropic scalar fields on the sphere

- **Some more analytical models:**

Covariance function	Angular power spectrum	Samples
$c(\psi)=\exp(\lambda(\cos\psi-1))\cdot J_0(\lambda\sin\psi)$ 	$C_n=4\pi\exp(-\lambda)\lambda^n/((2n+1)n!)$ 	
$c(\psi)=\exp(-\nu\psi)$ 	See Lantujoul et al. (2019) 	

J = Bessel function of the first kind

Random isotropic scalar fields on the sphere

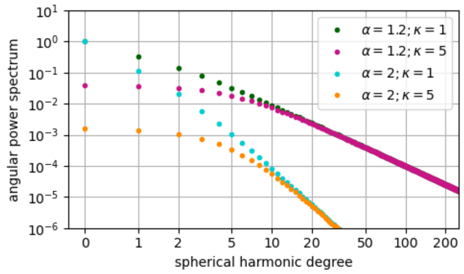
- **A versatile model defined through a stochastic partial differential equation (SPDE):**

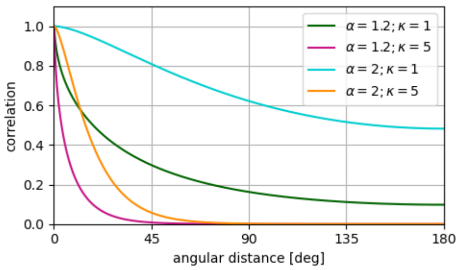
$$(\kappa^2 - \Delta)^{\alpha/2} \cdot f(\theta,\lambda) = \sigma \cdot w(\theta,\lambda) \quad \Rightarrow \quad C_n = \sigma^2 / (\kappa^2 + n(n+1))^{\alpha}$$

spherical Laplace operator

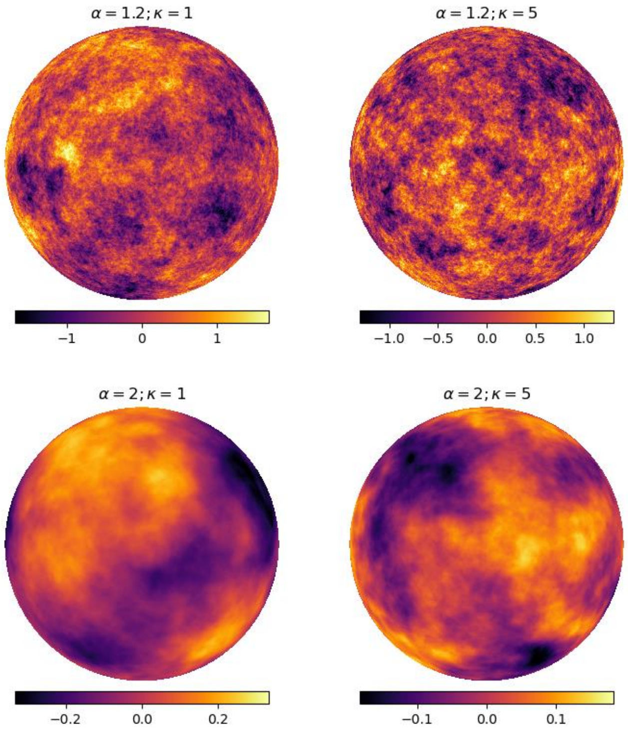
white noise on the sphere

  - Angular power spectrum is flat (white) at low degrees, then follows a power-law.
  - $\alpha/2$ = spectral index of the power-law
  - κ = cutoff degree





  - No analytical expression for the covariance
  - Analogous to the generalized Gauss-Markov (GGM) process of time (Langbein, 2004)



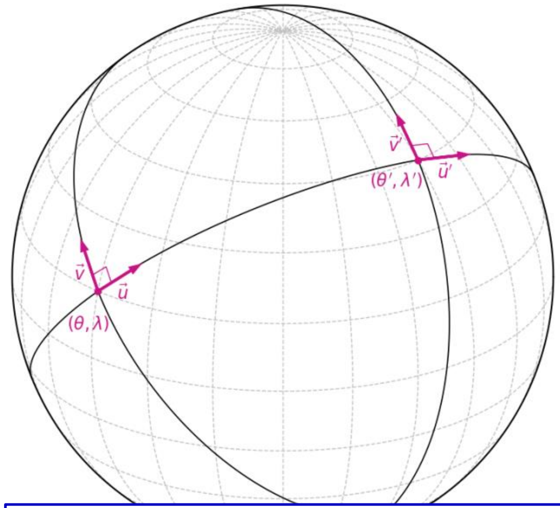
Random isotropic vector fields on the sphere

- **A random vector field on the sphere $\vec{f}(\theta,\lambda)$ is said to be (weakly) isotropic if, for any rotation R:**
  - $E[\vec{f}(R(\theta,\lambda))] = E[R(\vec{f}(\theta,\lambda))]$
  - $\mathrm{cov}[\vec{f}(R(\theta,\lambda)), \vec{f}(R(\theta',\lambda'))] = \mathrm{cov}[R(\vec{f}(\theta,\lambda)), R(\vec{f}(\theta',\lambda'))]$
  - **i.e., if its mean and covariance are invariant under rotations of the sphere simultaneously with the coordinate system.**

- **Isotropy implies:**

$$E\begin{bmatrix} f_r(\theta,\lambda) \\ f_\theta(\theta,\lambda) \\ f_\lambda(\theta,\lambda) \end{bmatrix} = \begin{bmatrix} \mu \\ 0 \\ 0 \end{bmatrix} \qquad \mathrm{cov}\left[\begin{bmatrix} f_u(\theta,\lambda) \\ f_v(\theta,\lambda) \\ f_r(\theta,\lambda) \end{bmatrix}, \begin{bmatrix} f_{u'}(\theta',\lambda') \\ f_{v'}(\theta',\lambda') \\ f_{r'}(\theta',\lambda') \end{bmatrix}\right] = \begin{bmatrix} c_{uu}(\psi) & c_{uv}(\psi) & c_{ur}(\psi) \\ c_{uv}(\psi) & c_{vv}(\psi) & c_{vr}(\psi) \\ -c_{ur}(\psi) & -c_{vr}(\psi) & c_{rr}(\psi) \end{bmatrix} = C(\psi)$$

The covariance of the field components <u>in the UVR frame</u> only depends on the distance between pairs of points.



The UVR frame of a pair of points
- U = along the great circle joining both points
- V = perpendicular to the great circle
- R = radial (not shown in the figure)

Besides, for a given point, and a pair of antipodal points:

$$\mathrm{var}\begin{bmatrix} f_r(\theta,\lambda) \\ f_\theta(\theta,\lambda) \\ f_\lambda(\theta,\lambda) \end{bmatrix} = \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_t^2 & 0 \\ 0 & 0 & \sigma_t^2 \end{bmatrix} \qquad \mathrm{cov}\left[\begin{bmatrix} f_r(\theta,\lambda) \\ f_\theta(\theta,\lambda) \\ f_\lambda(\theta,\lambda) \end{bmatrix}, \begin{bmatrix} f_r(\pi-\theta,\pi+\lambda) \\ f_\theta(\pi-\theta,\pi+\lambda) \\ f_\lambda(\pi-\theta,\pi+\lambda) \end{bmatrix}\right] = \begin{bmatrix} \gamma_r & 0 & 0 \\ 0 & \gamma_t & 0 \\ 0 & 0 & -\gamma_t \end{bmatrix}$$

Random isotropic vector fields on the sphere

- **A random isotropic vector field on the sphere has the following spectral decomposition:**

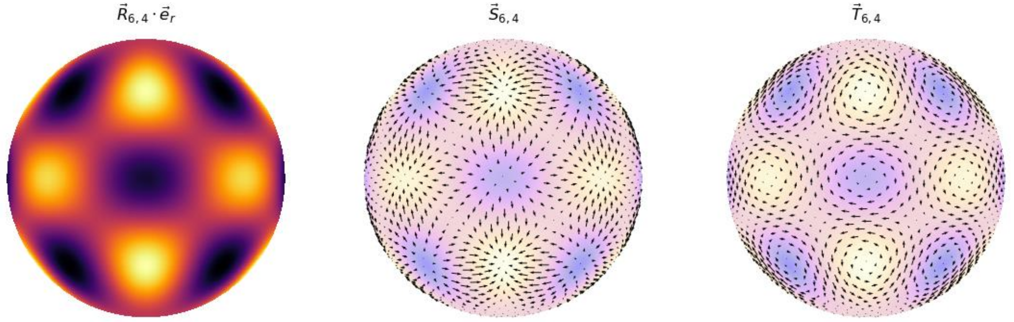
$$\vec{f}(\theta,\lambda) = \mu\vec{e}_r + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} r_{n,m}\vec{R}_{n,m}(\theta,\lambda) + s_{n,m}\vec{S}_{n,m}(\theta,\lambda) + t_{n,m}\vec{T}_{n,m}(\theta,\lambda)$$

– $\vec{R}_{n,m}, \vec{S}_{n,m}, \vec{T}_{n,m}$ = vector spherical harmonics = basis for vector fields on the sphere

– $\vec{R}_{n,m}(\theta,\lambda) = Y_{n,m}(\theta,\lambda)\cdot\vec{e}_r$ = basis for radial vector fields

– $\vec{S}_{n,m}(\theta,\lambda) = \vec{\nabla}Y_{n,m}(\theta,\lambda)/\sqrt{n(n+1)}$ = basis for tangential, curl-free (a.k.a spheroidal) vector fields

– $\vec{T}_{n,m}(\theta,\lambda) = \vec{\nabla}\times\vec{R}_{n,m}(\theta,\lambda)/\sqrt{n(n+1)}$ = basis for tangential, divergence-free (a.k.a toroidal) vector fields



– $r_{n,m}, s_{n,m}, t_{n,m}$ = spherical harmonic coefficient of the random vector fields = <u>random variables</u>

$$r_{n,m} = \iint\left(\vec{f}(\theta,\lambda) - \mu\vec{e}_r\right)\cdot\bar{\vec{R}}_{n,m}(\theta,\lambda)\sin\theta\, d\theta\, d\lambda \qquad s_{n,m} = \iint\left(\vec{f}(\theta,\lambda) - \mu\vec{e}_r\right)\cdot\bar{\vec{S}}_{n,m}(\theta,\lambda)\sin\theta\, d\theta\, d\lambda \qquad t_{n,m} = \iint\left(\vec{f}(\theta,\lambda) - \mu\vec{e}_r\right)\cdot\bar{\vec{T}}_{n,m}(\theta,\lambda)\sin\theta\, d\theta\, d\lambda$$

Random isotropic vector fields on the sphere

- The $r_{n,m}, s_{n,m}, t_{n,m}$ coefficient triplets satisfy:
  - $E[r_{n,m}, s_{n,m}, t_{n,m}] = [0, 0, 0]$
  - $[r_{n,-m}, s_{n,-m}, t_{n,-m}] = (-1)^m [\bar{r}_{n,m}, \bar{s}_{n,m}, \bar{t}_{n,m}]$
  - $$\text{cov}\left[\begin{bmatrix} r_{n,m} \\ s_{n,m} \\ t_{n,m} \end{bmatrix}, \begin{bmatrix} r_{n',m'} \\ s_{n',m'} \\ t_{n',m'} \end{bmatrix}\right] = \delta_{n,n'}\delta_{m,m'} \begin{bmatrix} c_n^{rr} & c_n^{rs} & c_n^{rt} \\ c_n^{rs} & c_n^{ss} & c_n^{st} \\ c_n^{rt} & c_n^{st} & c_n^{tt} \end{bmatrix} = \delta_{n,n'}\delta_{m,m'} C_n$$ ← The triplets are pairwise uncorrelated, and their covariance matrix only depends on the degree **n**.
  - If $\vec{f}(\theta,\lambda)$ is gaussian, the coefficient triplets of non-negative orders are not only uncorrelated, but also independent.

- The sequences $(c_n^{rr})_{0\le n\le\infty}$, $(c_n^{ss})_{0\le n\le\infty}$, $(c_n^{tt})_{0\le n\le\infty}$ can be called the radial, spheroidal & toroidal angular power spectra of the vector field.
  - They describe how the variance of the field is distributed across spherical harmonic degrees (i.e., spatial wavelengths) and across its radial, spheroidal and toroidal components.
  - Indeed: $\text{var}[f_r(\theta,\lambda)] = \sigma_r^2 = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} c_n^{rr}$ $\quad$ $\text{var}[f_\theta(\theta,\lambda)] = \text{var}[f_\lambda(\theta,\lambda)] = \sigma_t^2 = \sum_{n=0}^{\infty} \frac{2n+1}{8\pi} (c_n^{ss} + c_n^{tt})$

Random isotropic vector fields on the sphere

- The covariance function $C(\psi)$ and the angular power spectrum (C_n) of a random isotropic vector field on the sphere are linked via the following transform:

$$c_{rr}(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} c_n^{rr} P_{0,0}^n(\cos\psi) \qquad c_n^{rr} = 2\pi \int_0^{\pi} c_{rr}(\psi) \cdot P_{0,0}^n(\cos\psi) \cdot \sin\psi \, d\psi$$

$$c_{ur}(\psi) = -\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} c_n^{rs} P_{1,0}^n(\cos\psi) \qquad c_n^{rs} = -2\pi \int_0^{\pi} c_{ur}(\psi) \cdot P_{1,0}^n(\cos\psi) \cdot \sin\psi \, d\psi$$

$$c_{vr}(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} c_n^{rt} P_{1,0}^n(\cos\psi) \qquad c_n^{rt} = 2\pi \int_0^{\pi} c_{vr}(\psi) \cdot P_{1,0}^n(\cos\psi) \cdot \sin\psi \, d\psi$$

$$c_{uu}(\psi) + c_{vv}(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (c_n^{tt} + c_n^{ss}) P_{1,1}^n(\cos\psi) \qquad c_n^{tt} + c_n^{ss} = 2\pi \int_0^{\pi} (c_{uu}(\psi) + c_{vv}(\psi)) \cdot P_{1,1}^n(\cos\psi) \cdot \sin\psi \, d\psi$$

$$c_{uu}(\psi) - c_{vv}(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (c_n^{tt} - c_n^{ss}) P_{1,-1}^n(\cos\psi) \qquad c_n^{tt} - c_n^{ss} = 2\pi \int_0^{\pi} (c_{uu}(\psi) - c_{vv}(\psi)) \cdot P_{1,-1}^n(\cos\psi) \cdot \sin\psi \, d\psi$$

$$c_{uv}(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} c_n^{st} P_{1,-1}^n(\cos\psi) \qquad c_n^{st} = 2\pi \int_0^{\pi} c_{uv}(\psi) \cdot P_{1,-1}^n(\cos\psi) \cdot \sin\psi \, d\psi$$

  - The $P_{m,m'}^n$ are Gel'fand et al. (1958)'s generalized spherical functions. They are related to the Jacobi polynomials.

$$P_{0,0}^n(x) = P_n(x)$$

↑ Legendre polynomial

$$P_{1,0}^n(x) = P_n^1(x)/\sqrt{n(n+1)}$$

↑ Associated Legendre polynomial

$$P_{1,1}^n(x) = \left(\sqrt{1-x^2}\, dP_n^1/dx - P_n^1(x)/\sqrt{1-x^2}\right)/(n(n+1))$$

$$P_{1,-1}^n(x) = -\left(\sqrt{1-x^2}\, dP_n^1/dx + P_n^1(x)/\sqrt{1-x^2}\right)/(n(n+1))$$

Random isotropic vector fields on the sphere

- There is no (known) analytical {covariance/spectrum} couple for the tangential part.

- But the SPDE-based model previously introduced for scalar fields can be extended to vector fields:

$$(K^2 - \Delta \cdot \mathbf{I})^{A/2} \cdot D \cdot \vec{f}(\theta,\lambda) = \Sigma \cdot D \cdot \vec{w}(\theta,\lambda) \quad \Rightarrow \quad C_n = (K^2 + n(n+1) \cdot \mathbf{I})^{-A/2} \Sigma^2 (K^2 + n(n+1) \cdot \mathbf{I})^{-A/2}$$

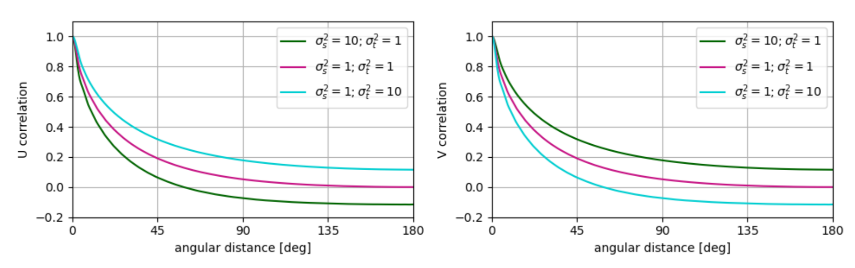
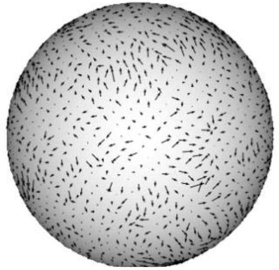
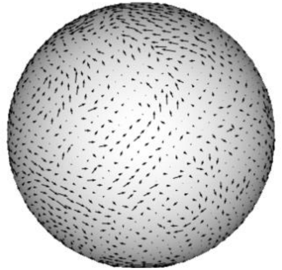
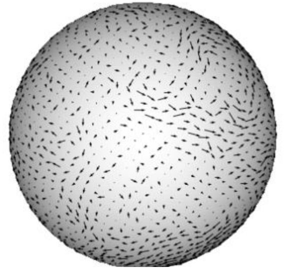
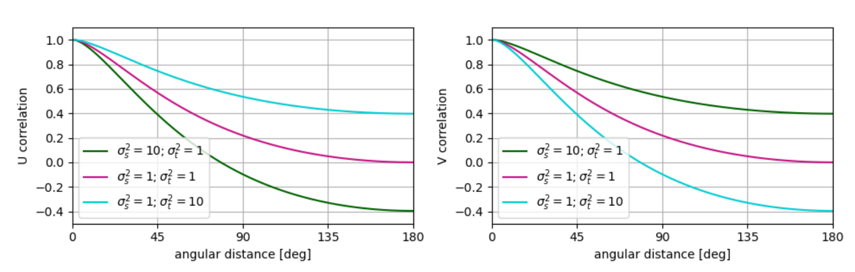
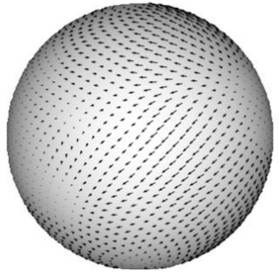
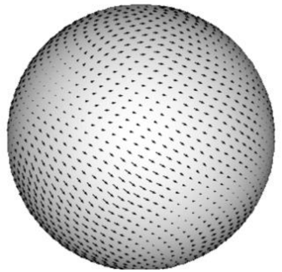
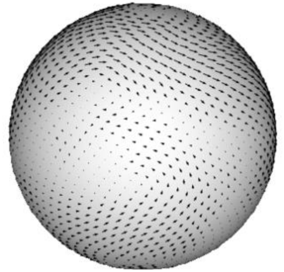
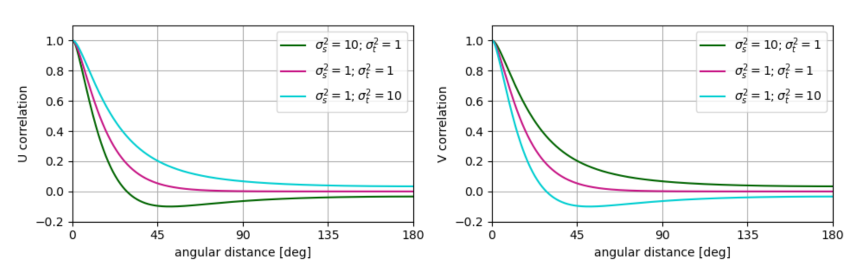
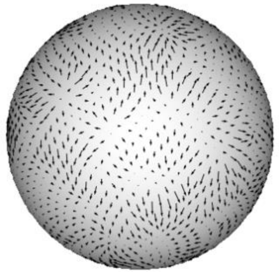
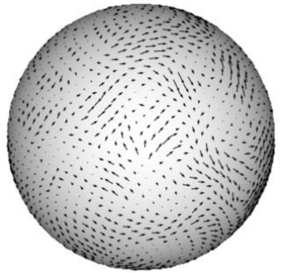
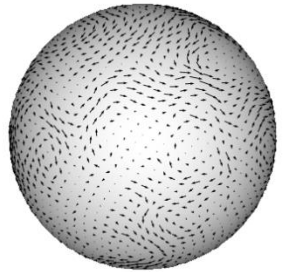
'(r,θ,λ) → (R,S,T)' differential operator

vector white noise on the sphere

  - A/2 = radial, spheroidal, toroidal spectral indices (3x3 symmetric positive-definite matrix)
  - K = radial, spheroidal, toroidal cutoff degrees (3x3 symmetric positive-definite matrix)
  - Σ = radial, spheroidal, toroidal standard deviations (3x3 symmetric positive-definite matrix)

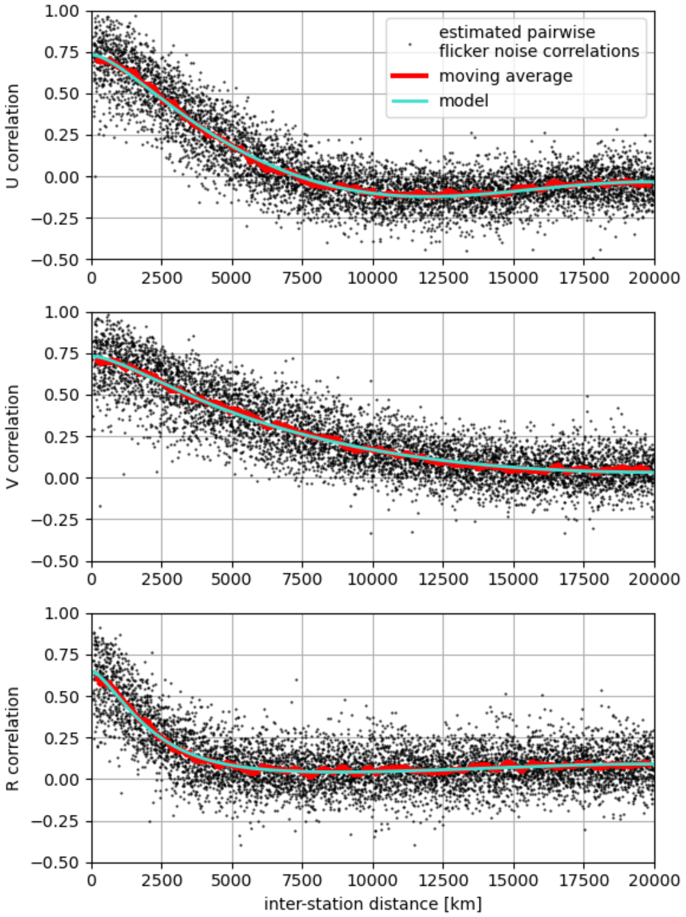
- If A, K and Σ are diagonal, then:
  - The (C_n) matrices are diagonal. → No correlations between the radial, spheroidal and toroidal components of the field.
  - $c_n^{rr} = \sigma_r^2 / (\kappa_r^2 + n(n+1))^{\alpha_r}$ $\quad c_n^{ss} = \sigma_s^2 / (\kappa_s^2 + n(n+1))^{\alpha_s}$ $\quad c_n^{tt} = \sigma_t^2 / (\kappa_t^2 + n(n+1))^{\alpha_t}$
  - C(ψ) is also diagonal. → No correlations between the U, V, R components of the field.
  - The radial component of the field can be treated independently as a scalar field.
    (This is why only tangential vector fields are shown in the next slide.)

Random isotropic vector fields on the sphere

	U & V correlation functions	$\sigma_s^2 = 10$; $\sigma_t^2 = 1$	$\sigma_s^2 = 1$; $\sigma_t^2 = 1$	$\sigma_s^2 = 1$; $\sigma_t^2 = 10$
$\alpha_s = \alpha_t = 1.2$ $K_s = K_t = 1$				
$\alpha_s = \alpha_t = 2$ $K_s = K_t = 1$				
$\alpha_s = \alpha_t = 2$ $K_s = K_t = 5$				

Application

- U/V/R <u>flicker</u> noise correlations estimated for > 12,000 pairs of GNSS station position time series
  - PPP time series from Nevada Geodetic Laboratory
  - See processing details in Gobron et al. (EGU23-3399)
- Long-range (> 50 km) correlations are well described by the previous SPDE-based model…
  - Different short-range correlation regime not shown here
  - See Gobron et al. (EGU23-3399)
- …plus extra variance at lowest (≤ 2) degrees.
  - Extra variance at degrees 0 and 1 may be explained by errors in alignment to reference frame.
  - Extra variance at degree 2 is puzzling…
- <u>White</u> noise in GNSS station position time series has different spatial correlation regimes.
  - See Gobron et al. (EGU23-3399)



Summary

- **A mathematical framework was set up to describe random isotropic vector fields on the sphere.**
  - **Their covariance is appropriately described in the rotationally invariant UVR frame.**
    (Not in the geocentric XYZ frame, nor in the topocentric East/North/Up frame.)
  - **They have a spectral representation in the vector spherical harmonic domain.**
  - **Their covariance C(ψ) and angular power spectrum (C_n) are linked through the integral transform on slide 13.**

- **This framework can be used to model, e.g., the spatial correlations of the 3D stochastic variations in GNSS station position time series, with potential applications in:**
  - **offset detection (Gazeaux et al. 2015),**
  - **velocity estimation (Benoist et al. 2020),**
  - **spatial filtering,**
  - **error budgets for GNSS velocity fields, plate rotation poles, strain maps…**

- **Applications may also be found in other domains involving vector fields on a sphere, e.g.:**
  - **winds,**
  - **ocean currents,**
  - **magnetic anomalies,**
  - **etc.**

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