High-Precision Calculating the Normal Height as the Coordinate Line's Length

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Abstract

In establishing a global system of normal heights, one of the key matters is the final choice of a height system to represent height marks. In addition to proving the advantages of the normal heights system, it is necessary to eliminate some "white spots" within itself. In 2004, a more accurate way of calculating normal heights as the length of a coordinate line in a spheroidal system was considered. In the meantime, the papers by some researchers contain only methods of "practically accurate" calculation of the orthometric height, which is associated with increasing knowledge of the earth's crust upper layers structure. At studying the normal height, it is required to develop methods of its high-precision calculation and explore the properties of various options for setting the corresponding curvilinear integral. An expression is obtained for the normal height as a segment of the coordinate line of the spheroidal system; the one obtained in 2004, which contained inaccuracies, was corrected. The proposed method can be applied at an arbitrary distance from the reference ellipsoid.

Introduction

Firstly, we must keep in mind, that the height systems were introduced **only** for the precision leveling processing, since all they are non-optimal for the water flow description. The main criteria for the orthometric and normal heights comparison are:

– behavior of heights on the earth's surface (1950-1970s) [4];
– behavior of heights inside the earth, see [1];
– behavior of heights at large and significant distances from the earth's surface (asymptotic properties) [2];
– behavior of heights on the level surfaces of reservoirs;
– possibility of theoretically exact calculation.

The orthometric system has the visible physical sense, but all their other properties are negative:

1. The impossibility of a practical accurate calculation (as the length of a segment of a real field line from the geoid to a point on the earth's surface).
2. The distribution of the real force of gravity along it is unknown.
3. The distribution of masses near and everywhere further is unknown (if it is known, then the problem becomes direct).
4. The location of the geoid is unknown (if known, there is no problem).
5. If the mass distribution is known, it is difficult to calculate volume integrals for each frame.
6. The orthometric heights difference is identical to the "vertical length" and does not reflect the behavior of the field with height.
7. If all this is overcome, then even in this case, the normal height has a number of advantages over the orthometric one.

As for normal heights, they do not have a simple visible representation, but they have the following advantages:

1. Possibility of practical (high) precision calculation:
– as the length of a segment of a normal field line from an ellipsoid to a point (Molodensky M. S., 1945)
– as normals to ellipsoid (usual way)
– as the spheroidal coordinate line length: [6], our work.
2. Low requirements for accuracy of gravimetric data.
3. Normal heights characterize level surfaces of a real field better than orthometric ones (normal heights are more constant on one level surface).

Disadvantages of normal heights:

1. Not constant on a level surface (reservoir), as dynamic.
2. Very different at a great distance when the normal field line is used (then we use the spheroidal coordinate line).

Problem Statement

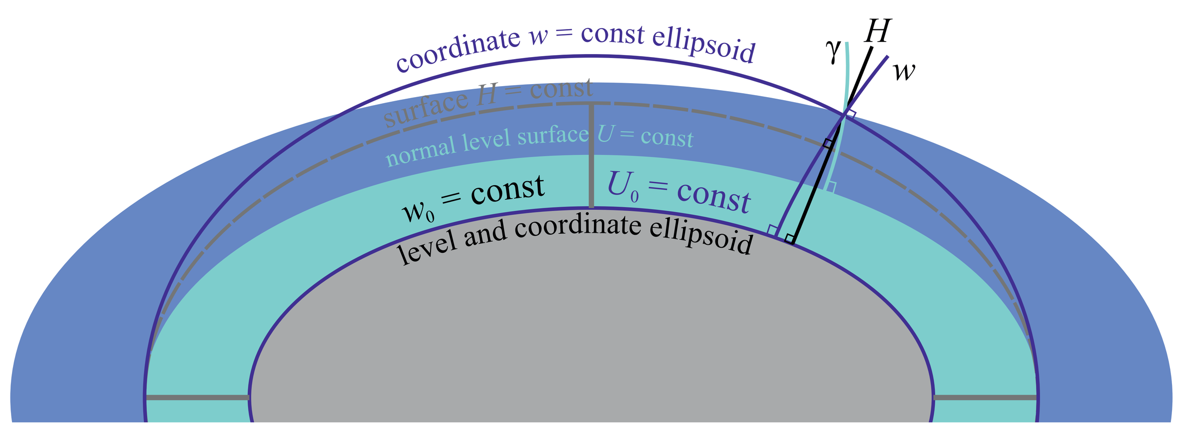
In the literature, there are discrepancies about the direction in which the telluroid points should be plotted from the ellipsoid for the subsequent calculation of the segment of normal height. There are three options (see Fig.):

– forceline of the normal field back (initial Molodensky's definition),
– coordinate line of the spheroidal system (some publications),
– normal to the ellipsoid.

Theoretically, the normal gravity field can be successfully used as an orthogonal coordinate system, since its force lines and level surfaces can serve as natural coordinate lines and coordinate surfaces. However, a normal force line does not have two characteristics that would be constant at each of its points with a change in only the third value, as in a conventional orthogonal coordinate system. The normal to the reference ellipsoid plays an important role in solving geometric problems of geodesy, but is of little use in physical matters. It is more convenient to use a curvilinear coordinate system associated with a family of ellipsoids confocal ($c = a \cdot e = \text{const}$) to the reference one with semiaxes a, b, especially since it contains closed expressions for the normal potential of gravity and all derivative elements. The method used so far for calculating the value of the normal height is based on the expansion of the normal gravity γ in a series using higher derivatives with respect to the geodetic coordinates at the point on the surface of the reference ellipsoid [5] with normal gravity γ_0, the expansion error naturally increases with distance from the ellipsoid. This Yeremeyev's formula is often considered as the definition of the normal height H^γ while it is only working formula [3]:

$$H^\gamma = \frac{1}{\gamma^m}\int_{(W_0)}^{(W)} g\,dh, \qquad \gamma^m = \frac{1}{H^\gamma}\int_{(U_0)}^{(U')} \gamma\,dH, \tag{1}$$

where $\gamma^m = \gamma_0 + \frac{H^\gamma}{2}\frac{\partial\gamma_0}{\partial H} + \frac{(H^\gamma)^2}{6}\frac{\partial^2\gamma_0}{\partial H^2} + \frac{(H^\gamma)^3}{24}\frac{\partial^3\gamma_0}{\partial H^3} + \dots$.



The normal height corresponds to the coordinate b or w in Molodensky's condition of the equality of the real $W_0 - W'$ and normal $U_0 - U'$ geopotential numbers:

$$W - W_0 = -\int g\,dh = U|_{u;b,w\Leftrightarrow H^\gamma} - U_0, \tag{2}$$

where g is the real gravity, dh is the measured elementary elevation, or

$$-\int g\,dh = \frac{GM}{c}\operatorname{arccot}\frac{b}{c} + \frac{\omega^2 a_0^2}{3q_0}\left[\left(\frac{3b^2}{2c^2}+\frac{1}{2}\right)\operatorname{arccot}\frac{b}{c} - \frac{3b}{2c}\right]P_2(\sin u) + \frac{\omega^2 a^2}{3}[1 - P_2(\sin u)] - \left(\frac{GM}{c}\operatorname{arccot}\frac{b_0}{c} + \frac{\omega^2 a_0^2}{3}\right), \tag{3}$$

here GM is geocentrical gravity constant, $c = a \cdot e$ is half of the focal disque, e being first eccentricity, ω is the angular velocity of the Earth's rotation, $q_0 = \left(\frac{3b^2}{2c^2}+\frac{1}{2}\right)\operatorname{arccot}\frac{c}{b} - \frac{3b}{2c} \approx 0,5\cdot 0,00015$.

For the first time, the question of the need to study and refine the method for calculating the normal height was raised by Miloš Pick and M. I. Yurkina in 2004 [7]. In their joint publication, the normal height is refined with respect to the gradient solution, taking into account the expression of the normal potential in the spheroidal system u, v, w (Niven's).

M. I. Yurkina in 2004 [6] gave a similar expression in the system u, L, b (Heiskanen-Moritz's), also indicating an explicit expression for the length of the segment of the coordinate line in the same system, however, the control calculations were not performed, so inaccuracies remained unnoticed in the proposed formulas for the auxiliary quantities, resulting in a low accuracy of the expression for the normal height H^γ.

1 Solution

1.1 The first approximation

First of all, we need to determine the main part of the normal height $H^\gamma_{2,3}$ using the Yeremeyev formula (1) from two or three approximations, then find a point with spheroidal coordinates u, b' or u, w' corresponding to the point on a telluroid with geodetic coordinates $B, H^\gamma_{2,3}$. A small error in the reduced latitude u due to ignorance of the exact spatial position of a point on the earth's surface will not affect the result in height (this can always be refined by successive aproximations), especially when using GNSS to determine leveling points, the problem disappears.

1.2 Refinement of the third spheroidal coordinates of the points on the telluroid

An inaccurate value of b' or w' will result in a mismatch between the left and right sides of the Molodensky's condition (2) or (3). Assuming that the left side is given, the right side can be expanded in a neighborhood of b' or w' and search for a small correction Δb or Δw in a linear approximation:

$$W - W_0 = -\int g\,dh = \left[U|_{u;b',w'\Leftrightarrow H^\gamma_{2,3}} + \frac{\partial U}{\partial b}\Delta b + \dots\right] - U_0,$$

where

$$\Delta b = \frac{[W - W_0] - [U|_{u;b',w'\Leftrightarrow H^\gamma_{2,3}} - U_0]}{\frac{\partial U}{\partial b}}.$$

Differentiating the right side of the Molodensky condition (3) with respect to the variable b or w, we get the derivative of $\frac{\partial U}{\partial b}$

$$\frac{\partial U}{\partial b} = -\frac{GM}{b'^2 + c^2} + \frac{\omega^2 a^2}{3q_0}\left[3\frac{b'}{c^2}\operatorname{arccot}\frac{b'}{c} - \left(\frac{3}{2}\frac{b'^2}{c^2} + \frac{1}{2}\right)\frac{\frac{1}{c}}{1 + \frac{b'^2}{c^2}} - \frac{3}{2c}\right] + \frac{2}{3}\omega^2 b'[1 - P_2(\sin u)],$$

which must be calculated at the point $u; b', w' \Leftrightarrow H^\gamma_{2,3}$. This expression of the derivative is more precise than that of M. I. Yurkina [6], where there was an excessive expansion of the arc.tangent.

As a result, we have a point on the telluroid with the third coordinate

$$b = b' + \Delta b, \qquad w = w' + \Delta w$$

for $b' = c\sinh w'$ and $b = c\sinh w$.

In this step, the precision control is performed, was obtained here up to 10^{-4} in the potential units $[m^2/s^2]$.

1.3 Evaluation of the curvilinear integral

To calculate the normal height as the length of the coordinate line w in the form of a curvilinear integral of the 1st kind

$$H^\gamma = \int_{w_0}^{w'} c\sqrt{\cosh w - \cos^2 u}\,dw,$$

where $c\sqrt{\cosh w - \cos^2 u} = h_w$ is the Lame coefficient of the third curvilinear coordinate w, expand the integrand $\sqrt{\cosh^2 w - \cos^2 u}$ into the Taylor series in the vicinity of the point with coordinates u, w_0 on the reference ellipsoid.

$$\begin{aligned}
H^\gamma = c\Bigg[\Bigg(&\sqrt{\cosh^2 w_0 - \cos^2 u} - \frac{\cosh^2 w_0}{\sqrt{\cosh^2 w_0 - \cos^2 u}} - \frac{\cosh^2 w_0\cos^2 u}{2\left(\sqrt{\cosh^2 w_0 - \cos^2 u}\right)^3} - \\
&- \frac{\cosh^4 w_0\cos^2 u}{2\left(\sqrt{\cosh^2 w_0 - \cos^2 u}\right)^5} - \frac{\cosh^2 w_0\cos^2 u(4\cosh^2 w_0 + \cos^2 u)}{8(\cosh^2 - \cos^2 u)^7}\Bigg)\int_{w_0}^{w'} 1\,dw + \\
&\left(\frac{\cosh w_0}{\sqrt{\cosh^2 w_0 - \cos^2 u}} + \frac{\cosh w_0\cos^2 u}{\left(\sqrt{\cosh^2 w_0 - \cos^2 u}\right)^3} + \frac{3}{2}\frac{\cosh^3 w_0\cos^2 u}{\left(\sqrt{\cosh^2 - \cos^2 u}\right)^5} + \right. \\
&\left. + \frac{1}{2}\frac{\cosh^3 w_0\cos^2 u}{(\cosh^2 w_0 - \cos^2 u)^7}\right)\int_{w_0}^{w'} \cosh w\,dw + \\
&\left(-\frac{\cos^2 u}{2\sqrt{\cosh^2 w_0 - \cos^2 u}} - \frac{3}{2}\frac{\cosh^2 w_0\cos^2 u}{\left(\sqrt{\cosh^2 w_0 - \cos^2 u}\right)^5} - \right. \\
&\left. - \frac{6}{8}\frac{\cosh^2 w_0\cos^2 u(4\cosh^2 w_0 + \cos^2 u)}{\left(\sqrt{\cosh^2 w_0 - \cos^2 u}\right)^7}\right)\int_{w_0}^{w'} \cosh^2 w\,dw + \\
&\left(\frac{1}{2}\frac{\cosh w_0\cos^2 u}{\left(\sqrt{\cosh^2 - \cos^2 u}\right)^5} + \frac{\cosh w_0\cos^2 u(4\cosh^2 w_0 + \cos^2 u)}{\left(\sqrt{\cosh^2 w_0 - \cos^2 u}\right)^7}\right)\int_{w_0}^{w'} \cosh^3 w\,dw \\
&- \frac{\cos^2 u}{8}\frac{4\cosh^2 w_0 + \cos^2 u}{\left(\sqrt{\cosh^2 w_0 - \cos^2 u}\right)^7}\int_{w_0}^{w'} \cosh^4 w\,dw + \dots\Bigg],
\end{aligned}$$

where there are table integrals:

$$\int \cosh^2 w\,dw = \frac{\sinh 2w}{2} + \frac{1}{2}w,$$
$$\int \cosh^3 w\,dw = \frac{\sinh^3 w}{3} + \sinh w,$$
$$\int \cosh^4 w\,dw = \frac{\sinh 4w}{32} + \frac{\sinh 2w}{4} + \frac{3}{8}w, \quad \text{etc.}$$

Here was the inaccuracy in the formula of M. I. Yurkina [6], expressed in u, L, b-system, this step is absent in the paper Jurkina M. I., Pick M. [7].

Conclusions

The possibility of the high-precision calculation of the normal height as the coordinate line of the spheroidal system is shown.

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