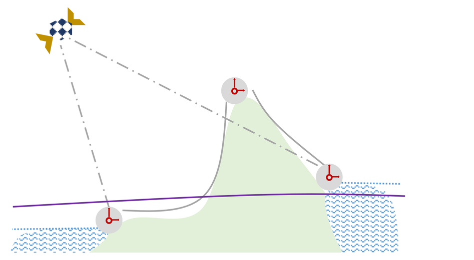
# Unification of Height Systems using Chronometric Geodesy - A More Realistic Scenario

Asha Vincent and Jürgen Müller
Institut für Erdmessung (IfE), Leibniz Universität Hannover, Germany

## 1 Chronometric levelling

- Optical clocks are sensitive to gravity field variations according to Einstein's theory of relativity
- The ticking rate of a clock lifted by 1 cm on Earth surface changes by a factor of $10^{-18}$
- Frequency comparisons between high-performance clocks in networks enable chronometric leveling for establishing a unified global height system



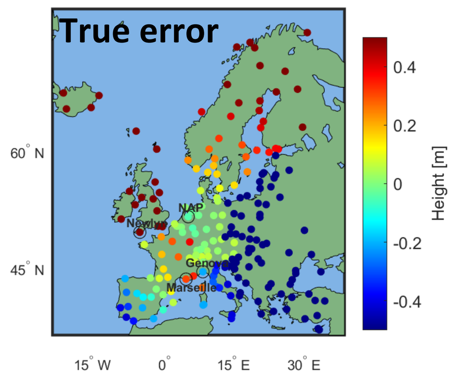
$$\frac{\Delta f}{f} = \frac{W_B - W_A}{c^2} = \frac{C_A - C_B}{c^2}$$

Chronometric levelling

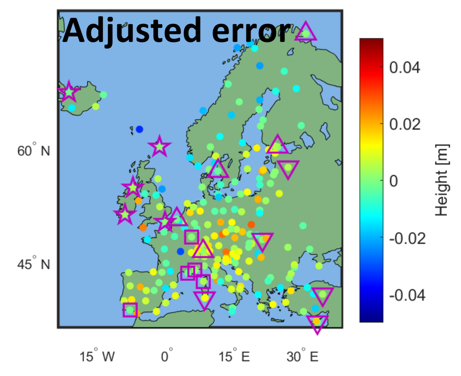
## 3 Estimation of complex errors in local height systems

### Static errors in local height systems (LHS)

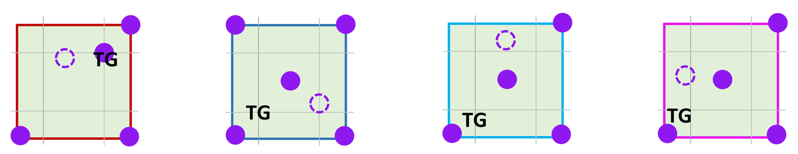
- Height equation : $H^L_i = H^U_i + a^L\Delta X + b^L\Delta Y + c^L + t^L\Delta S + m^L(H^L_i/500)$ — used 5 clocks in each LHS
- The datum of the re-unified system is assumed to be the datum of ③ ($c^L = 0$) — (①□ ②☆ ③△ ④▽)



True error (a, b, c, t, m) between the local height systems and the a-priori height system before unification (left). Residual error between the unified height systems and a-priori height system after clock-based unification (right)



- The RMS error of the re-unified system with the a-priori system is of the order of ~1 cm and the standard deviations of the errors estimated reach maximum values for $c^L$ of 3-4 cm
- A further goal is to optimize the number of clocks and their spatial distribution



Spatial distribution of clocks in each LHS

- For estimating tilts, clock sites should be selected at points which correspond to max. and min. values of each tilt
- For estimating offsets, a clock site that is least affected by systematic tilts is important

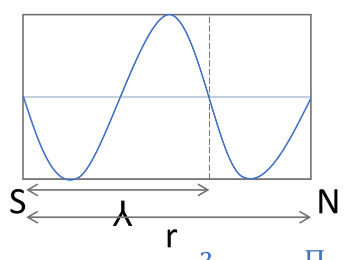
### DHHN92 vs. DHHN2016 (German Main Height Network) - Elevation changes between new realizations

- Assuming we have clock observations at the two epochs

1) $C^{92}_i = C^{16}_i - \Delta C_i$    2) $\Delta W_{ji}^{16} - \Delta W_{ji}^{92} = \Delta C_j - \Delta C_i$

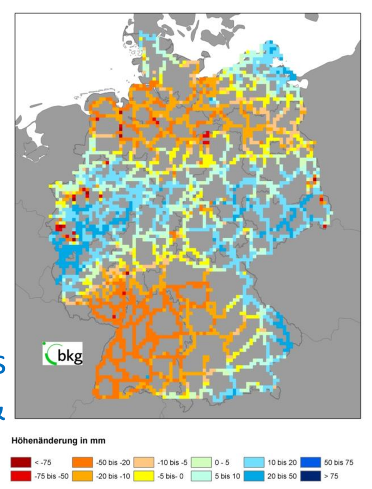
- As a simple case (or in a more general way), we just assumed the variation as a cosine wave from south to north:

$$\Delta C_i = A\cos(k\, r_i + \emptyset)$$



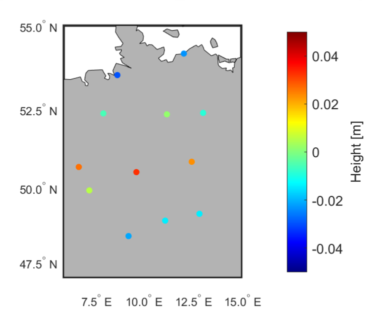


$A \sim 3$ cm, $\lambda \sim \frac{2}{3} r$, $\emptyset \sim \frac{\Pi}{2}$



Height differences between DHHN92& DHHN2016

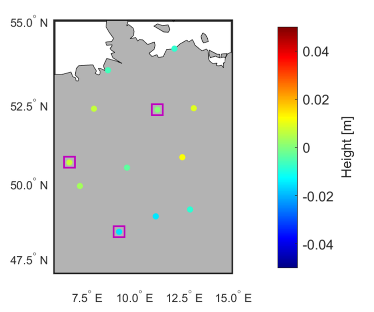
- Repeated regression by changing wavelength:



Assumed error

Estimated parameters of the periodic function after clock based adjustment

| Wave length (m) | Phase (rad) | Amplitude (m) | RMS (m) |
|---|---|---|---|
| 5.930667e+05 | 1.5128 | 0.0366 | 0.0090 |



Adjusted error

## 2 Height system unification

Height system unification involves the estimation of complex errors (systematic tilts and biases, etc.) between the local height systems

### Estimation of errors in height systems

Closed loop simulations using clock-based adjustment for estimating latitudinal tilt (a), longitudinal tilt (b), offset (c), tide gauge tilt (t), mountain tilt (m), noisy leveling line tilt (n), etc.

$$H^L_i = \frac{C^U_i}{\gamma_i} + biases + RN$$

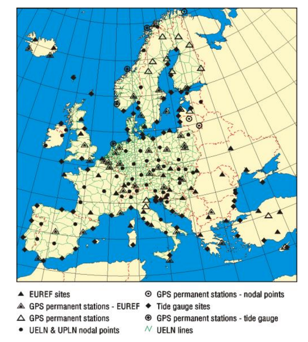
$H^L_i$ ⇨ Height of the leveling point in the local system
$H^U_i$ ⇨ Height of the leveling point in the unified system

$$\Delta W_{ij} = W_i^U - W_j^U = -(C_i^U - C_j^U) + RN$$

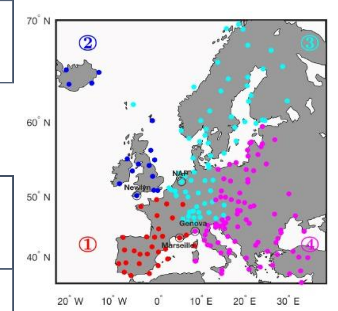
Clock observation equation between clock sites i and j in terms of geopotential number (C)

### Methodology

**A priori system – EUVN/2000 (European Vertical Reference Network)**
Classification
⇩
**Noise added to local height values $H_i^L$ (offset, tilts & random noise)**
Random noise added to clock observations
⇩
**Reunification (Clock-based adjustment)**
Comparison (reunified system with the a priori system)



EUVN network



Classified height systems

### Tidal correction on clock observations

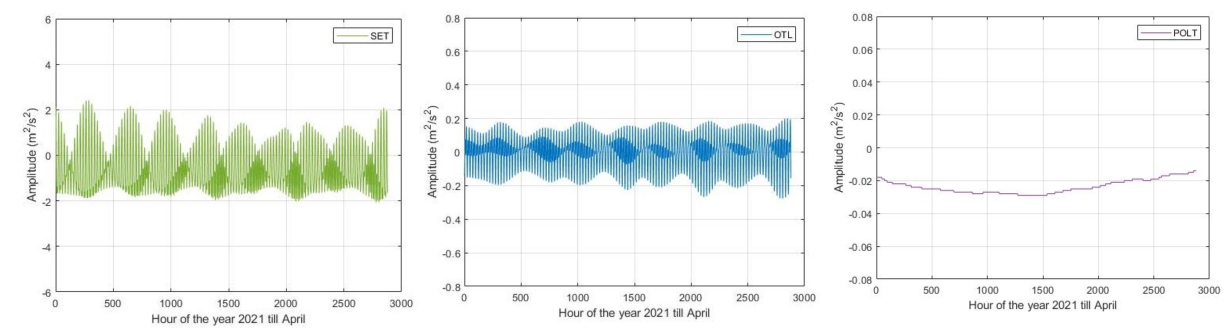
- In real scenarios, the clock observations are affected by various tidal effects such as solid earth tide (SET), ocean tide loading (OTL), pole tide (POL), etc.
- As clocks rest on the deformable earth surface, the effective potential variation due to mass change and corresponding surface deformation has to be considered

$$\Delta W_{ij} = W_i^U - W_j^U = -(C_i^U - C_j^U) + RN - (\Delta C_i(t) - \Delta C_j(t))$$

$\Delta C_i(t)$ is the summed tidal effects at clock location i

- Model errors applied for realistic simulation of clock observations

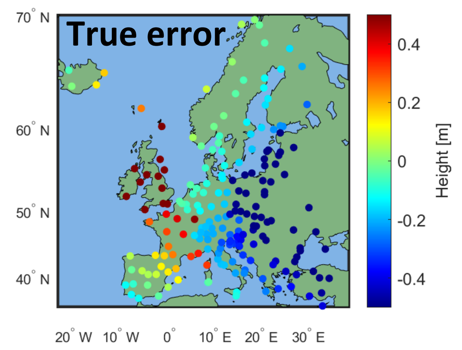
## 4 Tidal correction on clock observations



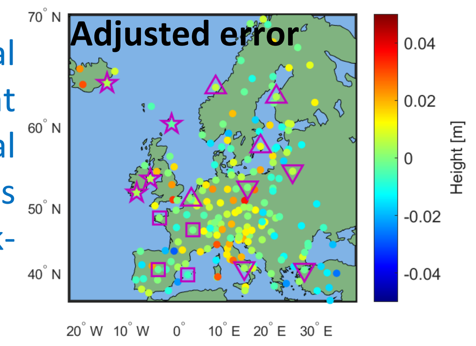
Time series of SET (ETERNA), OTL (SPOTL) and POL (ETERNA) tide at clock site near PTB

### Tidal correction applied with (10%) model error

Height equation: $H^L_i = H^U_i + a^L\Delta X + b^L\Delta Y + c^L$ — used 4 clocks in each LHS



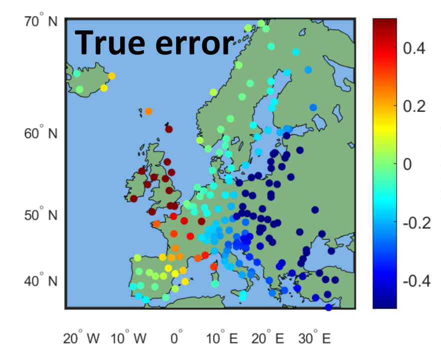
True error (a, b, c) between the local height systems and the a-priori height system before unification (left). Residual error between the unified height systems and a-priori height system after clock-based unification (right)



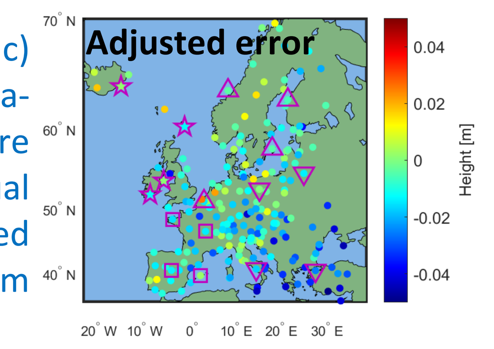
## 5 LHS with different frequency standards

- Through a weighted adjustment, the unification can still be improved, even when two clocks in each LHS have an uncertainty of only $10^{-17}$ (~ 0.1 m)

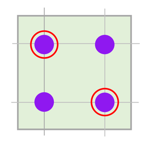
Height equation: $H^L_i = H^U_i + a^L\Delta X + b^L\Delta Y + c^L$) — used 4 clocks in each LHS



True error (a, b, c) between the LHS & the a-priori system before unification (left). Residual error between the unified systems & a-priori system after unification (right)



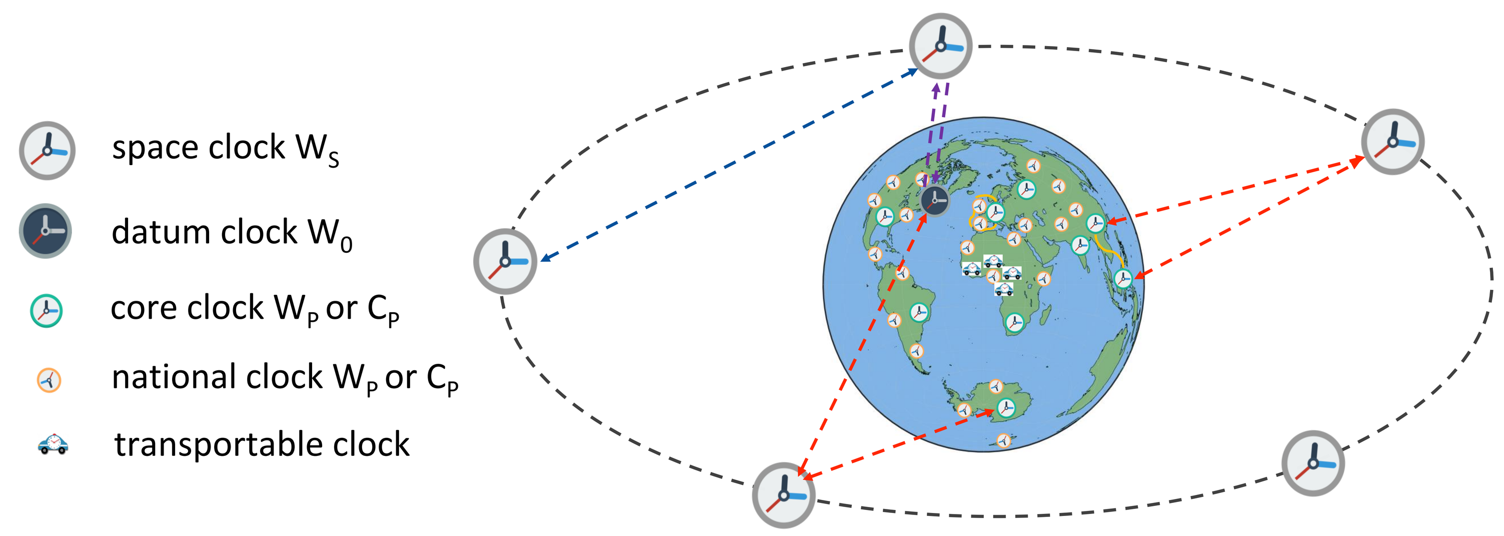
- The accuracy of bias estimation depends on spatial distribution of good clocks



Spatial distribution of good clocks and worse clocks (red) in each LHS

## 6 Future perspectives

Clocks are expected for the realization of an international height reference system.





A hybrid clock network (different types of clocks as well as various frequency link techniques) for the realization of an international height reference system.

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## References

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Asha Vincent
{vincent@ife.uni-hannover.de}