Politecnico di Torino
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Department of Structural, Geotechnical and Building Engineering

EGU General Assembly 2023

Towards a general constitutive model for snow

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Vienna, April 24th, 2023

Outlines

1. Introduction
2. The model
3. Numerical implementation
4. Conclusions

1. Introduction: bibliography and state-of-the-art of snow models

Soil mechanics models

Cam Clay

- *Roscoe & Schofield (1963)*
- *Nova et al. (2003)*
- …

Mod. Cam Clay

- *Roscoe & Burland (1968)*
- *Ortiz & Pandolfi (2004)*
- …

Solid mechanics models

- Overstress theory (Perzyna, 1963)
- Advanced yield functions (Bigoni & Piccolroaz, 2010; Panteghini & Lagioia, 2017)
- …

Snow models

Mod. Cam Clay (for snow)

?

Hansen & Brown (1988)

Meschke et al. (1996)

Cresseri et al. (2010) → Sintering model (rounded and dry snow)

Gaume et al. (2018) → Anticrack model (faceted snow)

…

Lack of a general model

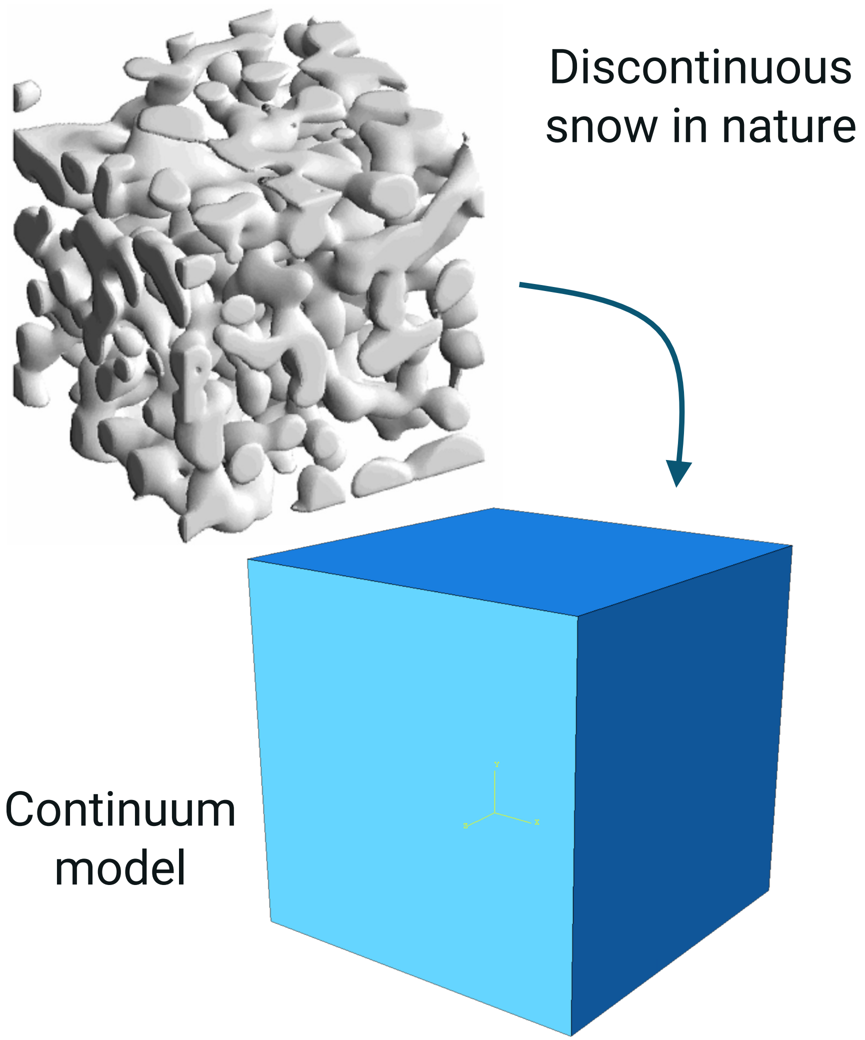
2. The model: general aspects

- My initial proposal for the model is based on **three key points**:

  1. the general framework of the elastic-visco-plastic model proposed by **Cresseri & Jommi (2005)**
  2. the **overstress theory of Perzyna (1963)**, accounting for irrecoverable strains even inside the yield locus
  3. A **new formulation for the yield surface**

- Initial hypotheses and assumptions: **small strains, continuity, homogeneity, and isotropy**

- The **temperature is constant** during the test time (purely mechanical model)



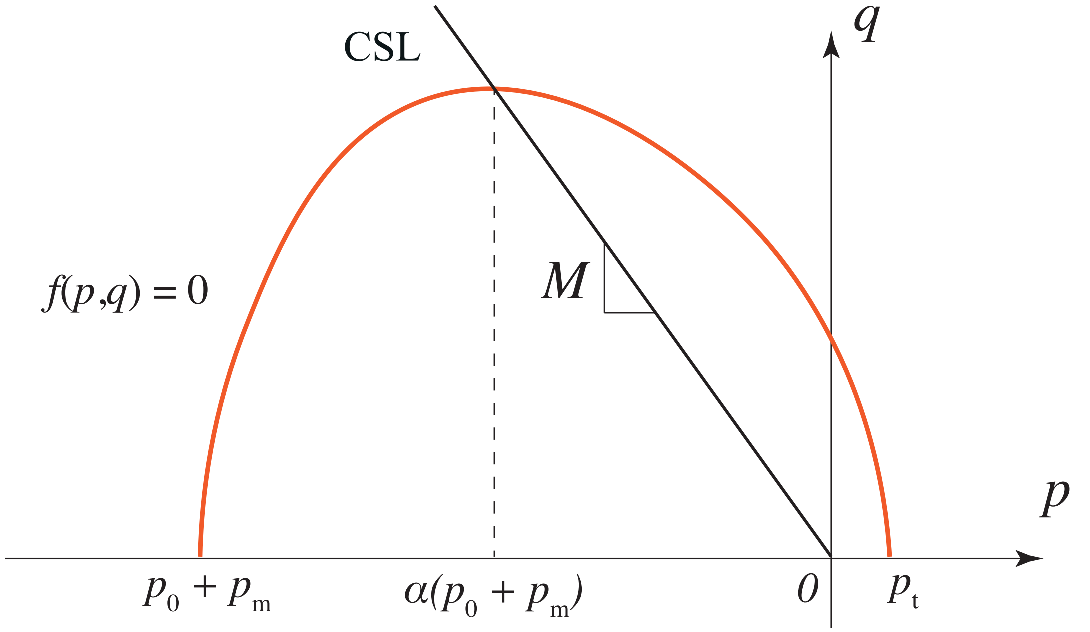
2. The model: yield surface (i)

The model uses an **improved yield surface** that was obtained starting from the Mod. Cam Clay for snow (Cresseri & Jommi, 2005) and the Panteghini & Lagioia (2017) methodology to deform the yield surface

$$f(p,q) = \frac{1}{p_{atm}^2}\left\{q^2 - 4\alpha^2 M^2 (p_0 + p_m)^3 \frac{\phi_1}{\phi_2^2}\right\}$$

$$\phi_1 = (\alpha - 1)(p - p_t)(p + p_0 + p_m)[p_t + \alpha(p_0 + p_m)]$$

$$\phi_2 = -p(p_0 + p_m - p_t) + 2p(p_0 + p_m)\alpha + (p_0 + p_m)[-(\alpha - 2)p_t + (p_0 + p_m)\alpha]$$



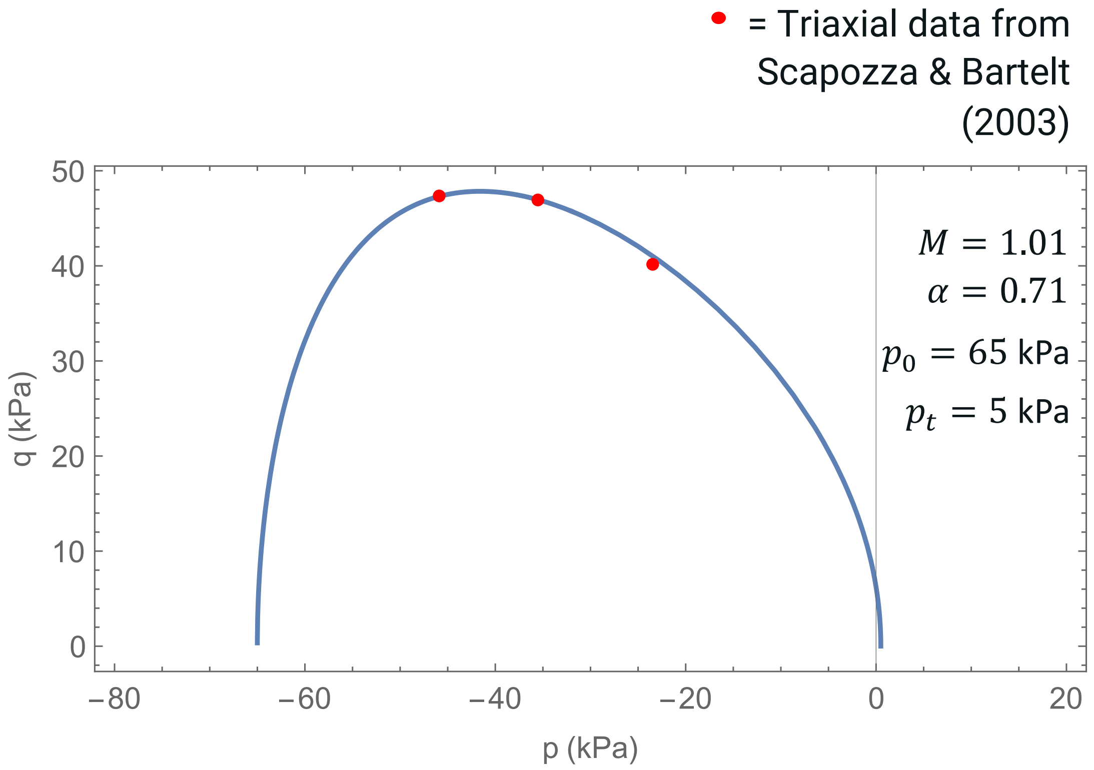
The new surface accounts for the **additional strength in compression** (p_m) and **tension** (p_t) due to sintering

2. The model: yield surface (ii)

The **meridian section of the yield surface** is described by the following function:

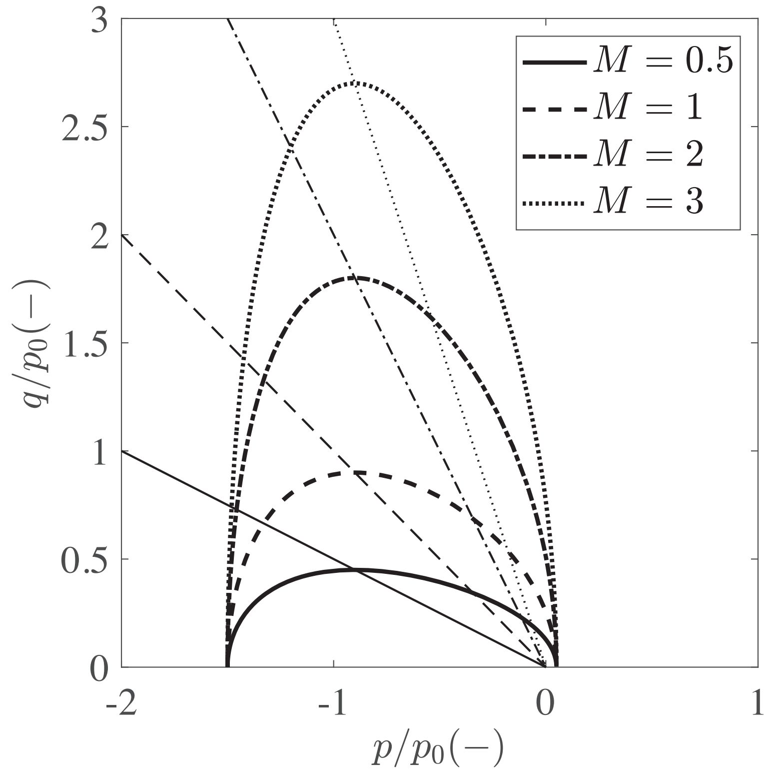
$$f(p,q) = \frac{1}{p_{atm}^2}\left\{q^2 - 4\alpha^2 M^2 (p_0 + p_m)^3 \, \frac{\phi_1}{\phi_2^2}\right\}$$

- The function describes a surface which is **simply convex and smooth** at any point of the p-q space
- Experimental findings (e.g., Scapozza & Bartelt, 2003) suggest that a asymmetric yield surface is best suited for snow
- The surface can potentially adapt to various snow conditions (e.g. different grain types, complex stress-paths, etc.)

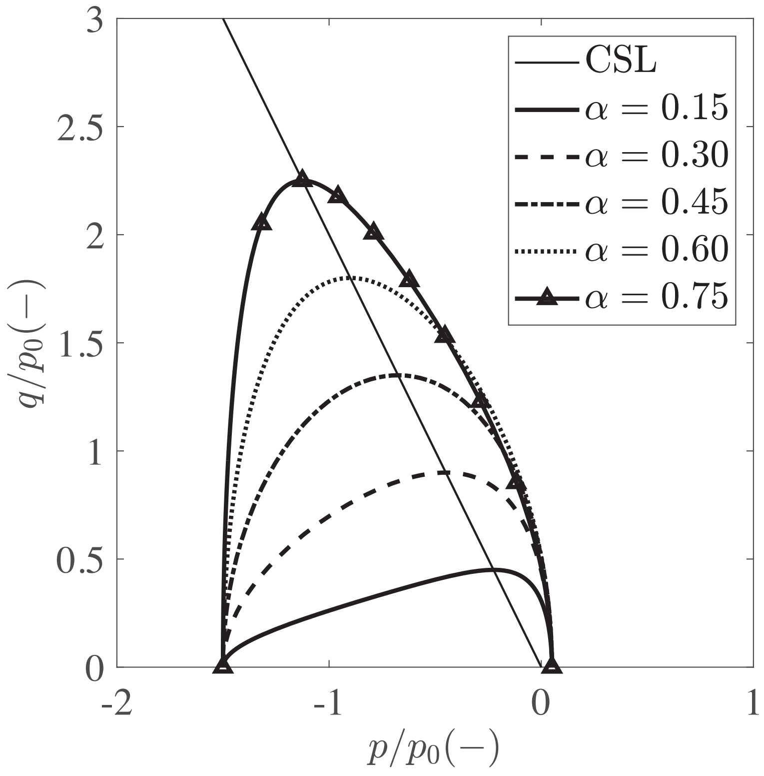


2. The model: yield surface (iii)

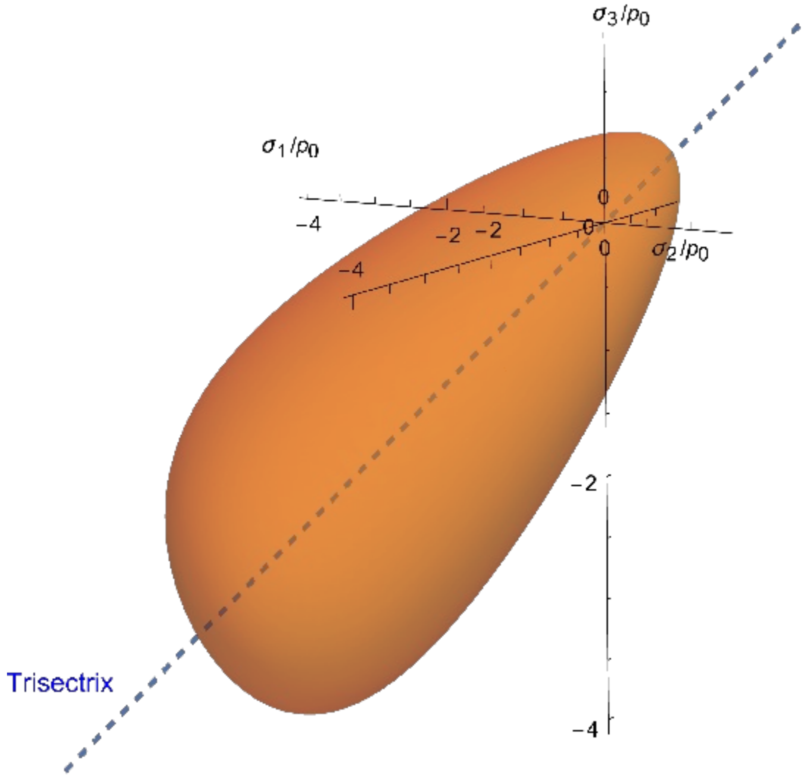
Effect of the shape **parameter M** on the $f = 0$ curve



Effect of the shape **parameter α** on the $f = 0$ curve



3D view in the Haigh-Westergaard stress space



2. The model: visco-plastic strain potential

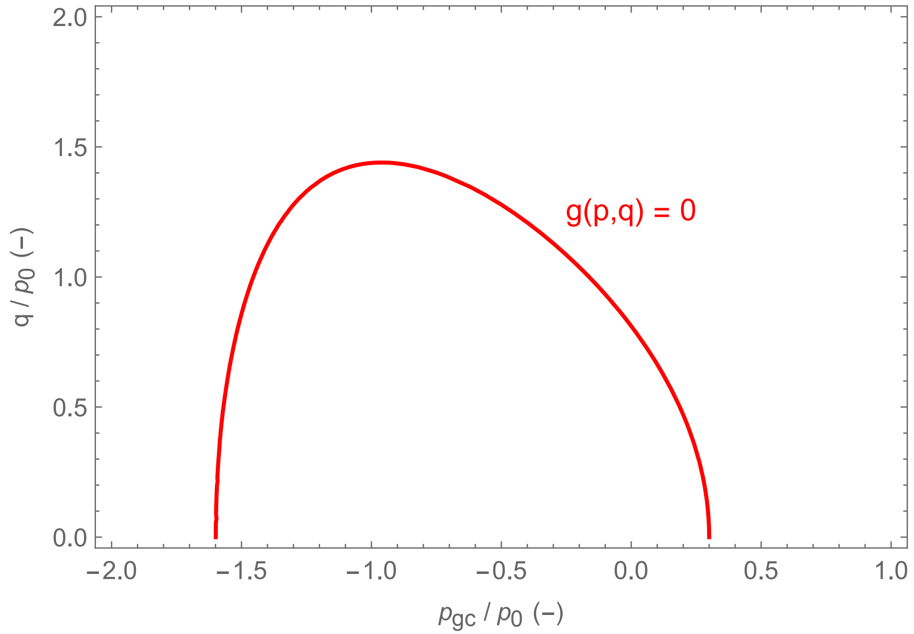
The irreversible strain potential has a mathematical expression quite similar to the yield function

$$g(p,q) = q^2 - 4\alpha^2 M^2 p_{g0}^3 \frac{\phi_{g1}}{\phi_{g2}^2}$$

$$\phi_{g1} = (\alpha - 1)(p - p_{gt})(p + p_{g0})(p_{gt} + \alpha p_{g0})$$

$$\phi_{g2} = -p(p_{g0} - p_{gt}) + 2pp_{g0}\alpha + p_{g0}[-(\alpha - 2)p_{gt} + p_{g0}\alpha]$$

$g(p,q) = 0$ describes a curve passing always through the stress point



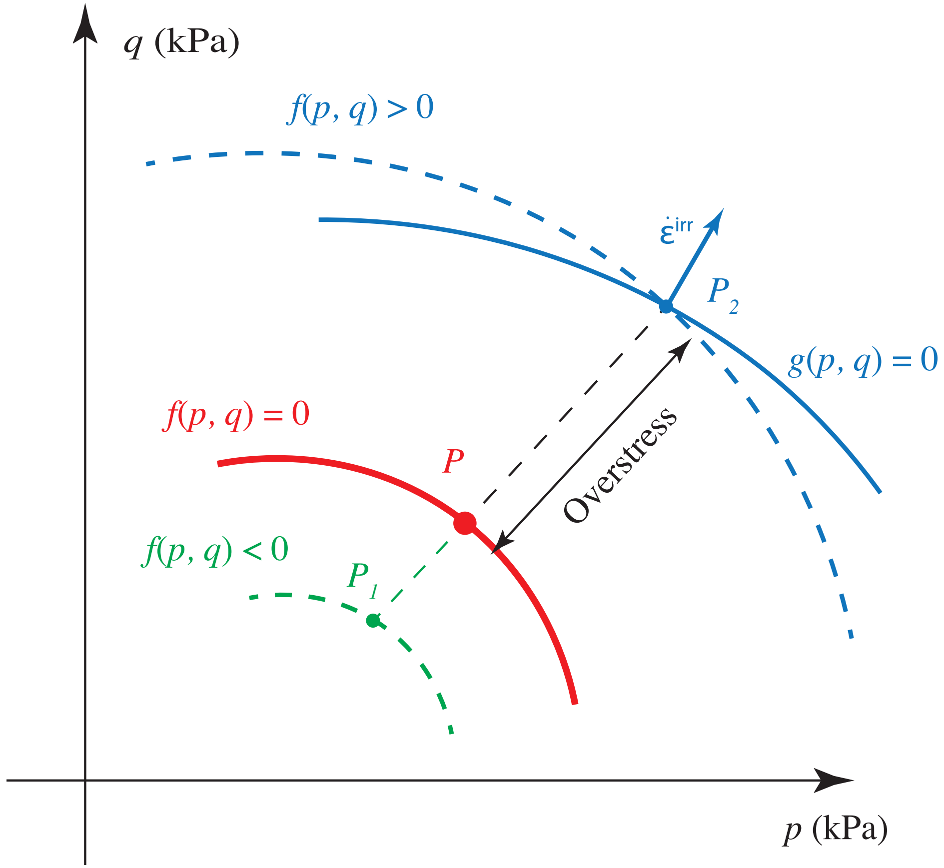
Parameters

$p_0 = 50$ kPa
$p_m = 5$ kPa

$M = 1.5$
$\alpha = 0.6$

2. The model: flow rule

A **non-associative flow rule of the Perzyna type** is considered to take into account the presence of viscous effects even inside the elastic region (according to Cresseri et al., 2010)



$$\dot{\boldsymbol{\epsilon}}^{\mathrm{irr}} = \bar{\gamma}\, \phi(f) \frac{\partial g}{\partial \boldsymbol{\sigma}} \frac{1}{|\nabla g|}$$

Component	Aspect	Description
Fluidity parameter (always positive)	$\bar{\gamma} = \dfrac{\psi\sqrt{p^2+q^2}}{\sqrt{3}\,p_0}$	Distance of the stress point from the origin
Viscous nucleus (always positive)	$\phi(f) = e^{af}, a > 0$	Component of the deformation velocity, function of the overstress
Normal unit vector	$\dfrac{\partial g}{\partial \boldsymbol{\sigma}} \dfrac{1}{\lvert\nabla g\rvert}$	direction of the irreversible strains

2. The model: hardening and sintering laws

- **Isotropic hardening** rule typical of the Mod. Cam Clay with the addition of the hardening parameter ξ

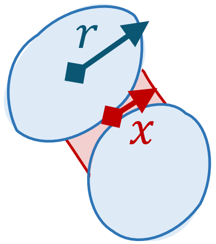
$$\dot{p}_0 = -\xi \frac{v}{\lambda - \kappa} p_0 \dot{\epsilon}_v^{\text{irr}}$$

- The rate of variation of p_m is expressed as:

$$\dot{p}_m = \pi_m b_{max} \dot{S}$$

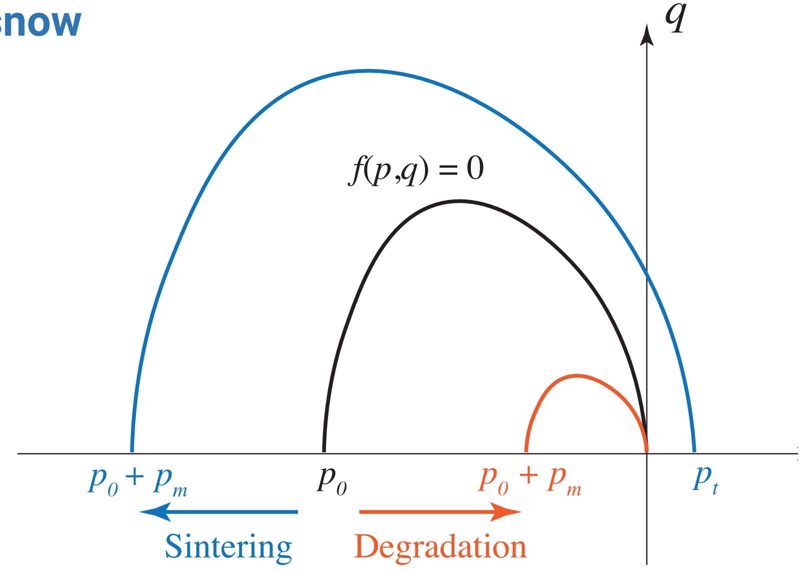
π_m = constitutive parameter
b_{max} = maximum ratio between the bonding necks and the radius of the particles

$$b = \frac{x}{r}$$



- Sintering law for snow by Cresseri et al. (2010) describing the current amout of sintering S

$$S = \underbrace{\tilde{S}_0(t_s, r, T)}_{\tilde{S}_0 = \text{amount of sintering in the unstressed snow}} \underbrace{\left[1 - \tanh\left(C \int_0^t \sqrt{\left(\dot{\epsilon}_v^{\text{irr}}\right)^2 + \left(\dot{\epsilon}_{dev}^{\text{irr}}\right)^2}\right)\right]}_{\text{Degradation term}}$$



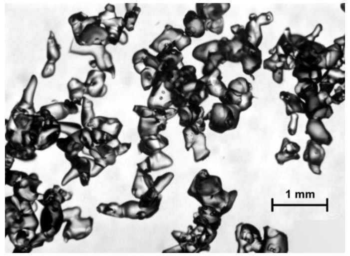
2. The model: parameters

The model is based on **13 parameters** that can be obtained from laboratory tests, observation of the snow grains, literature data, etc.

Parameters	Type	Test for validation
κ, λ, G	Elastic	Triaxial tests, shear tests
M, α, χ, χ_g	Plastic (yield locus)	Shear tests, Compression 1D tests, literature data
ψ, a	Viscous	Compression tests, triaxial tests, relaxation and creep tests, literature data
ξ	Hardening	Literature data, snow grain observation
C, π_m, ω	Sintering	Sintering tests, literature data

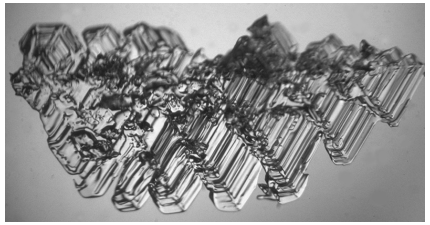
For round snow (hard slab):

$$\chi = \chi_g = 0.05 \div 0.1$$
$$\omega = 0.05$$
$$\xi = 1$$



For faceted snow (weak layer):

$$\chi = \chi_g = 0$$
$$\omega = 0$$
$$\xi = 0$$

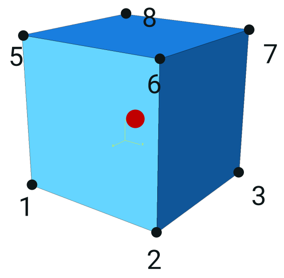


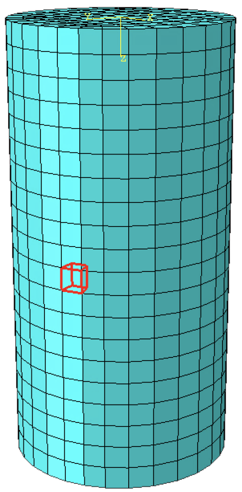
3. Numerical implementation

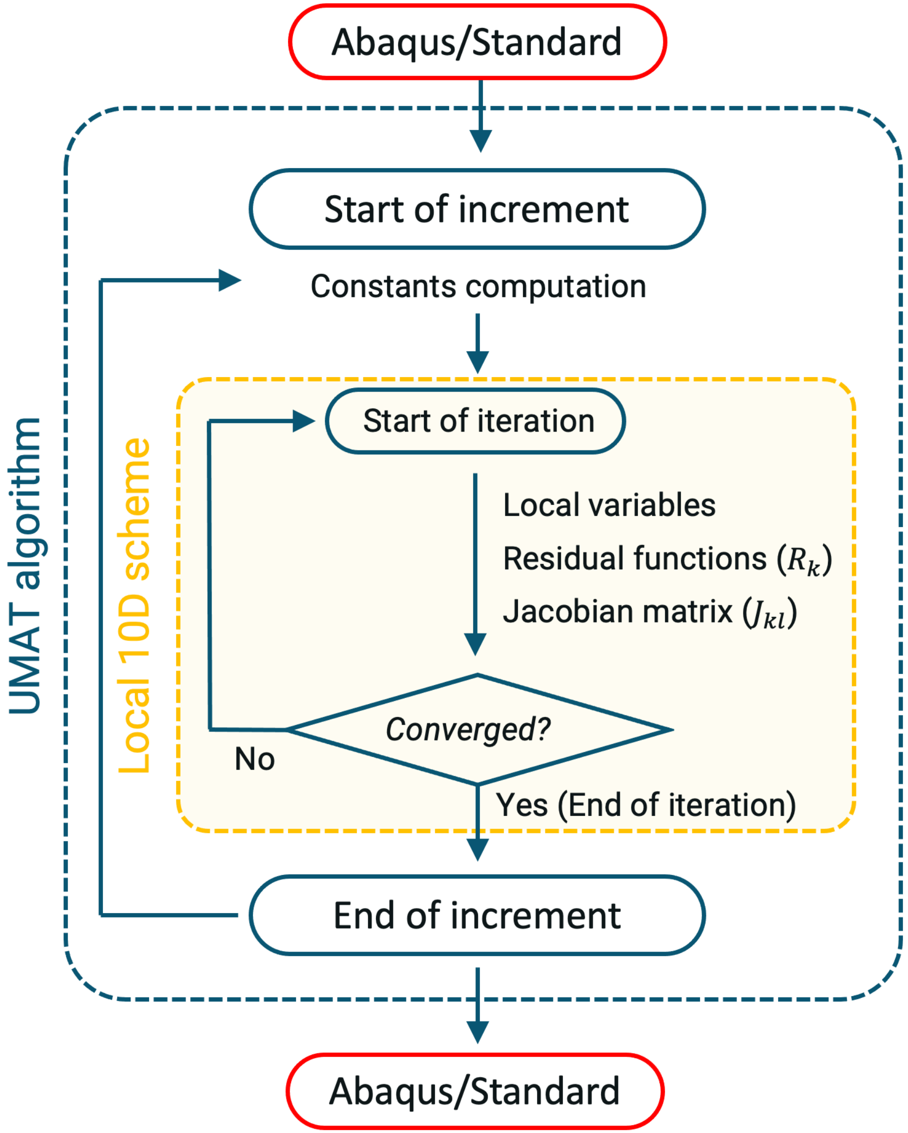
The model was time-integrated following a **fully implicit backward Euler method scheme**

The **system of 10 non-linear differential constitutive equations** is solved with a local Powell hybrid method (a generalized Newton-Raphson method)

The model has been implemented into the **UMAT format (Fortran 77) for the Abaqus/Standard** Finite Element code







3. Numerical implementation: the 10D discretized system

$$R_{1to6} = \frac{1}{Z_1}\left(\Delta\boldsymbol{\sigma} - \boldsymbol{D}^e(\boldsymbol{\sigma}_{n+1})\Delta\boldsymbol{\epsilon} + \boldsymbol{D}^e(\boldsymbol{\sigma}_{n+1})\,\beta_{n+1}\,\frac{\partial g}{\partial\boldsymbol{\sigma}}\frac{1}{|\nabla g|}\bigg|_{n+1}\Delta t\right) = 0$$

$$R_7 = \frac{1}{Z_2}\left(g\left(\boldsymbol{\sigma}_{n+1}, p_{g0}\right)\right) = 0$$

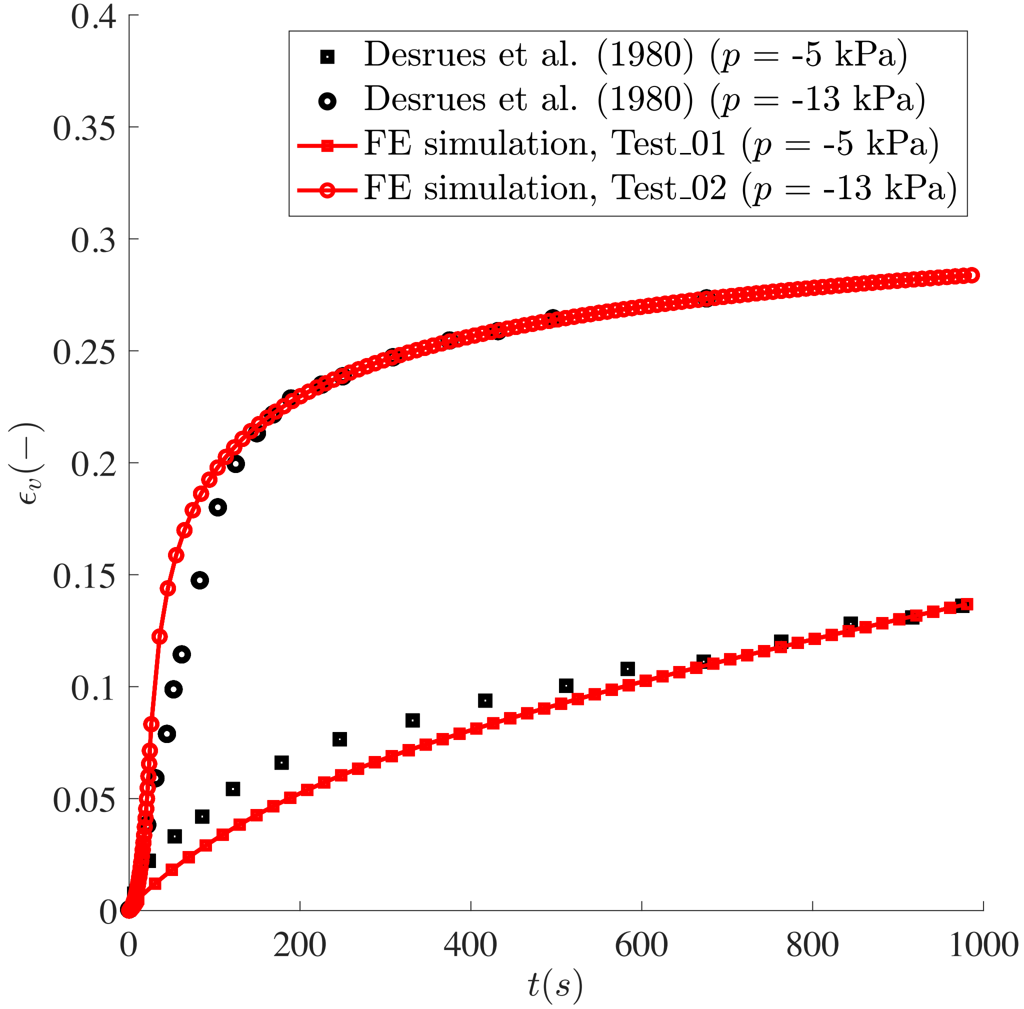
$$R_8 = \frac{1}{Z_1}\left(\Delta p_0 + \xi\,\frac{v_{n+1}}{\lambda - k}(p_{0n} + \Delta p_0)\Delta\epsilon_v^{\mathrm{irr}}\right) = 0$$

$$\mathrm{R}_9 = \Delta p_m - \pi_m b_{max}\Delta S = 0$$

$$R_{10} = S_n + \Delta \mathrm{S} - \tanh(\omega t_s)\left\{1 - \tanh\left[C\left(\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_i+\Delta t_i}\sqrt{\left(\frac{\epsilon_{v_i}^{\mathrm{irr}}}{\Delta \mathrm{t}_i}\right)^2 + \left(\frac{\epsilon_{dev_i}^{\mathrm{irr}}}{\Delta \mathrm{t}_i}\right)^2}\,dt\right) + \int_{t_n}^{t_{n+1}}\sqrt{\left(\frac{\epsilon_v^{\mathrm{irr}}}{\Delta t}\right)^2 + \left(\frac{\epsilon_{dev}^{\mathrm{irr}}}{\Delta t}\right)^2}\,dt\right)\right]\right\} = 0$$

3. Numerical implementation: some results

Volumetric creep (Desrues et al., 1980)



Initial conditions

Test	p^0 **(kPa)**	p_0 **(kPa)**	v_0 **(-)**	T **(°C)**	r_0 **(mm)**
Test_01	0.0	2	4.58	-5	0.2
Test_02	0.0	2	4.58	-5	0.2

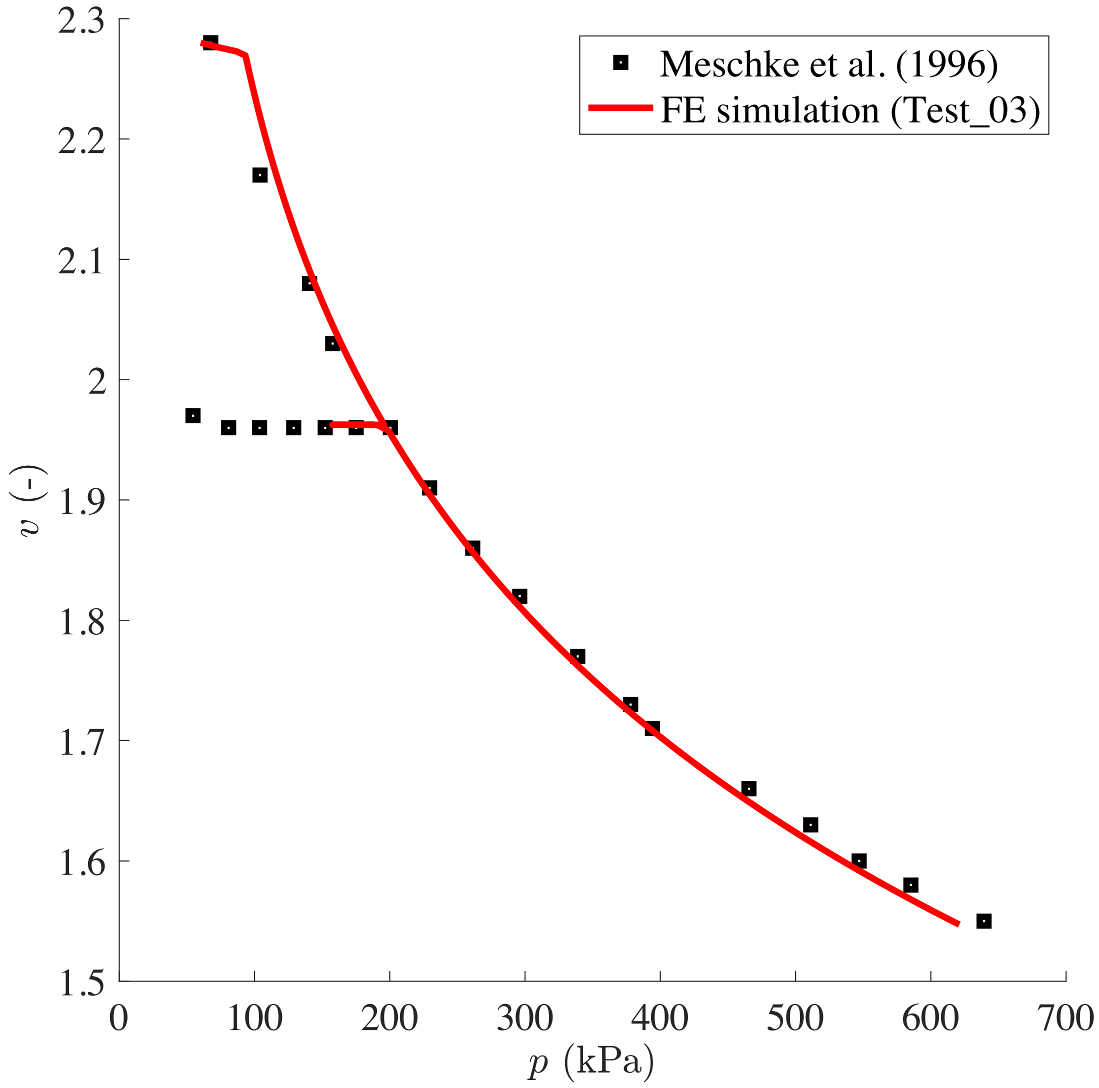
Model parameters

Test	λ **(-)**	k **(-)**	G **(kPa)**	ψ **(-)**	a **(-)**	π_m **(-)**
Test_01 & Test_02	0.35	0.02	2114	1.2e-4	16	40

Test	χ (-)	C (-)	M (-)	α (-)	ξ (-)	ω (-)
Test_01 & Test_02	0.05	0.01	2.88	0.475	1	0.05

3. Numerical implementation: some results

Volumetric compression (Meschke et al., 1996)



Initial conditions

Test	p^0 (kPa)	p_0 (kPa)	v_0 (-)	T (°C)	r_0 (mm)
Test_03	-60.0	77	2.28	-5	0.2

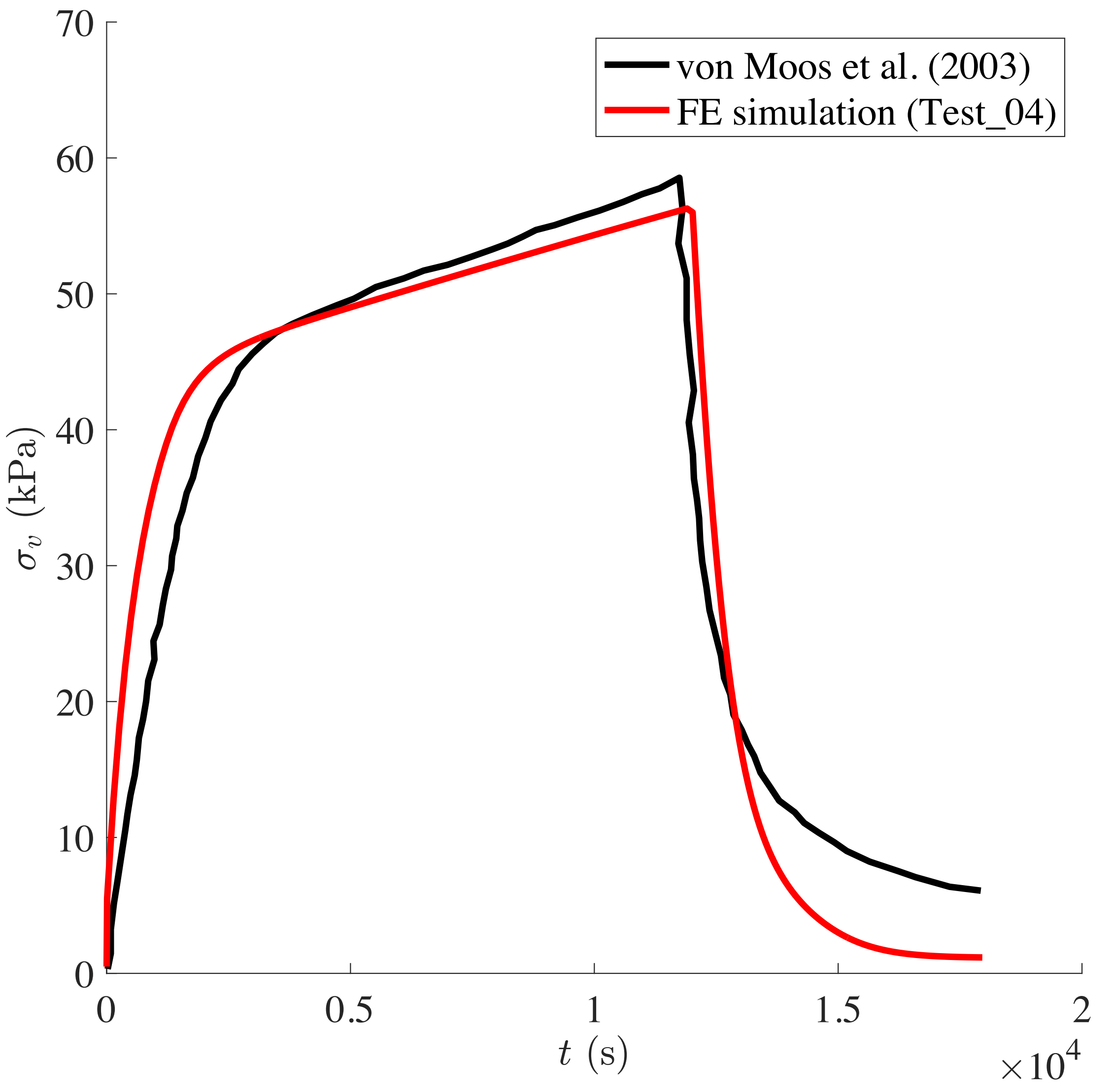
Model parameters

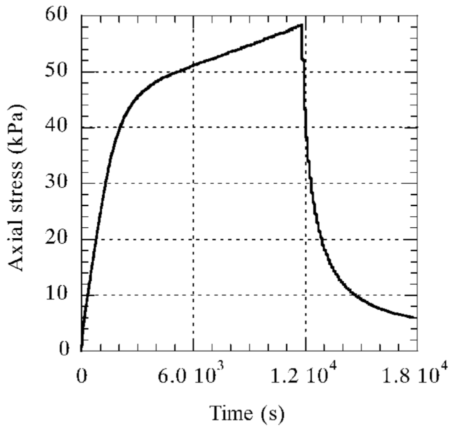
Test	λ (-)	k (-)	G (kPa)	ψ (-)	a (-)	π_m (-)
Test_03	0.35	0.02	12000	2.0e-7	16	40

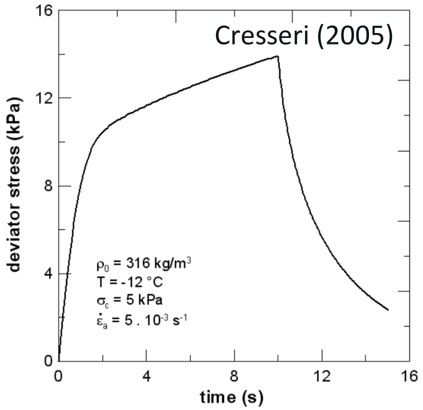
Test	χ (-)	C (-)	M (-)	α (-)	ξ (-)	ω (-)
Test_03	0.05	0.01	2.88	0.475	1	0.05

3. Numerical implementation: some results

Triaxial compression – long time (von Moos et al., 2003)







Previsious FE simulation – only qualitative (Cresseri, 2005)

Initial conditions

Test	p^0 **(kPa)**	p_0 **(kPa)**	v_0 **(-)**	T **(°C)**	r_0 **(mm)**
Test_04	0.0	25	2.90	-12	0.118

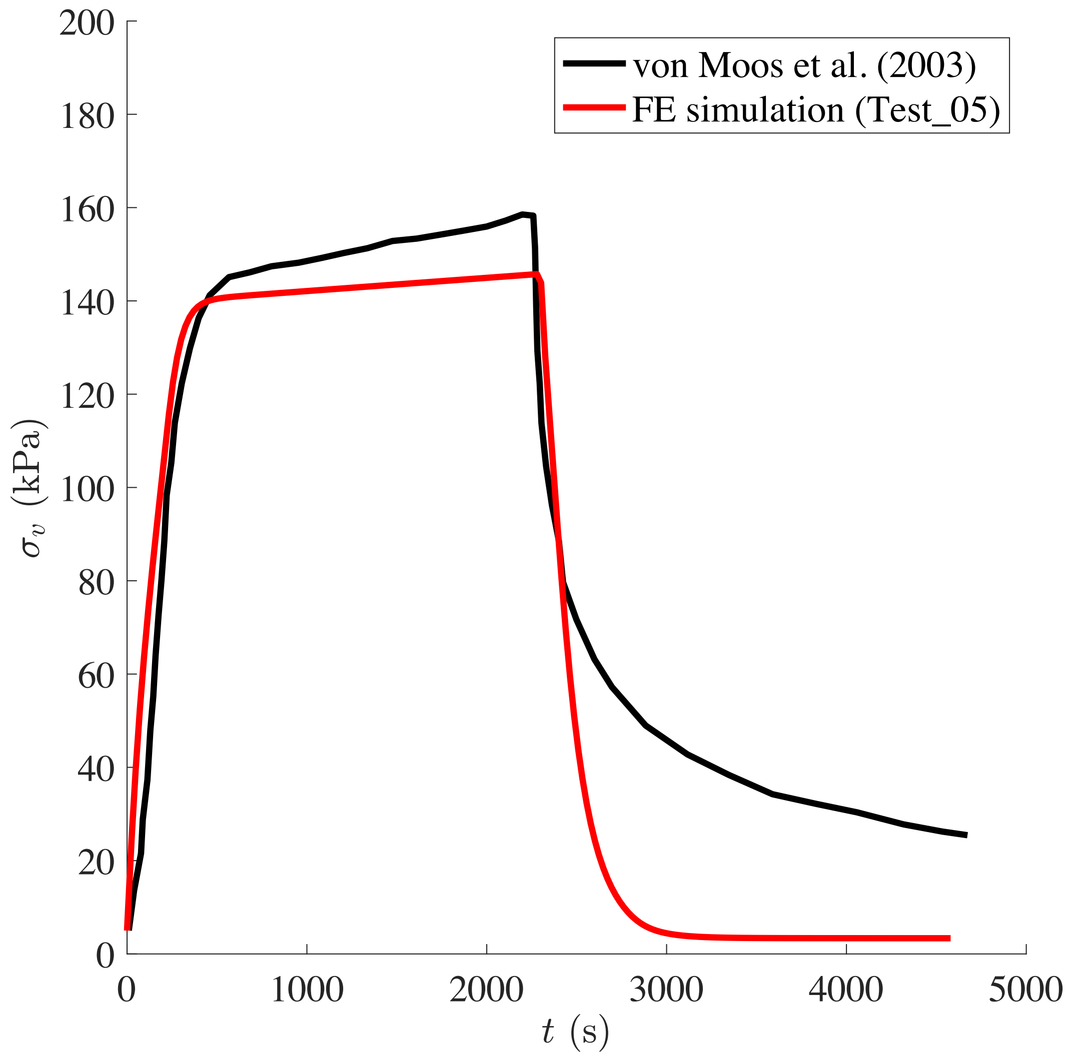
Model parameters

Test	λ **(-)**	k **(-)**	G **(kPa)**	ψ **(-)**	a **(-)**	π_m **(-)**
Test_04	0.35	0.02	8000	4.2e-6	0.35	40

Test	χ **(-)**	C **(-)**	M **(-)**	α **(-)**	ξ **(-)**	ω **(-)**
Test_04	0.05	0.01	2.88	0.475	1	0.05

3. Numerical implementation: some results

Triaxial compression – short time (von Moos et al., 2003)



Initial conditions

Test	p^0 (kPa)	p_0 (kPa)	v_0 (-)	T (°C)	r_0 (mm)
Test_05	-5.0	100	2.44	-12	0.118

Model parameters

Test	$\boldsymbol{\lambda}$ **(-)**	$\boldsymbol{k}$ **(-)**	$\boldsymbol{G}$ **(kPa)**	$\boldsymbol{\psi}$ **(-)**	$\boldsymbol{a}$ **(-)**	$\boldsymbol{\pi_m}$ **(-)**
Test_05	0.35	0.02	20000	2.0e-5	0.35	40

Test	$\boldsymbol{\chi}$ **(-)**	$\boldsymbol{C}$ **(-)**	$\boldsymbol{M}$ **(-)**	$\boldsymbol{\alpha}$ **(-)**	$\boldsymbol{\xi}$ **(-)**	$\boldsymbol{\omega}$ **(-)**
Test_05	0.05	0.01	2.88	0.475	1	0.05

4. Conclusions

- The model is a **generalization and an improvement of an existing snow model** (Cresseri & Jommi, 2005)
- The model reproduces satisfactorily different features of the mechanical behavior of snow
- The model is in **good agreement** (especially from a quantitative point of view) with many lab findings
- The model can reproduce some tests better than existing snow models
- **Possible further developments**:
  i. Consideration of finite strains
  ii. Definition of specific testing procedures for the identification of model parameters
  iii. Execution of testing campaigns to extend the available data for parameter estimation

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Thank you for your kind attention!

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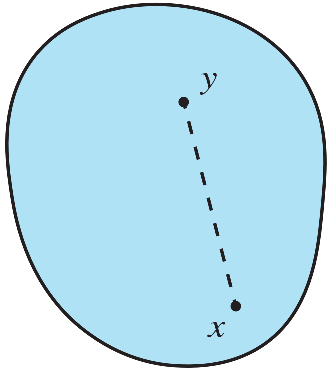
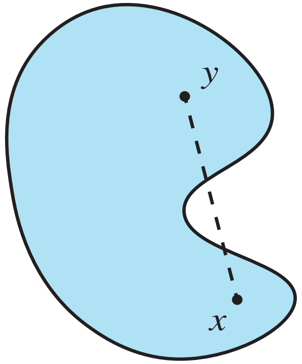
2. The model: convexity (i)

Convexity is a fundamental requirement for the yield surface to guarantee the stability of the model with respect to arbitrary stress and strain paths



Convex set



Non-convex set

A scalar-valued function $f(\boldsymbol{\sigma}, p_c)$ of the stress tensor $\boldsymbol{\sigma} \in D$, where D is a convex subset of $\mathbb{R}^6$, and of the hidden variable $p_c \in \mathbb{R}^+$, is **quasi-convex** if all its lower contour sets:

$$L_f(f_0) = \{(\boldsymbol{\sigma} \in D, p_c \in \mathbb{R}^+) | f(\boldsymbol{\sigma}, p_c) \leq f_0)\}$$

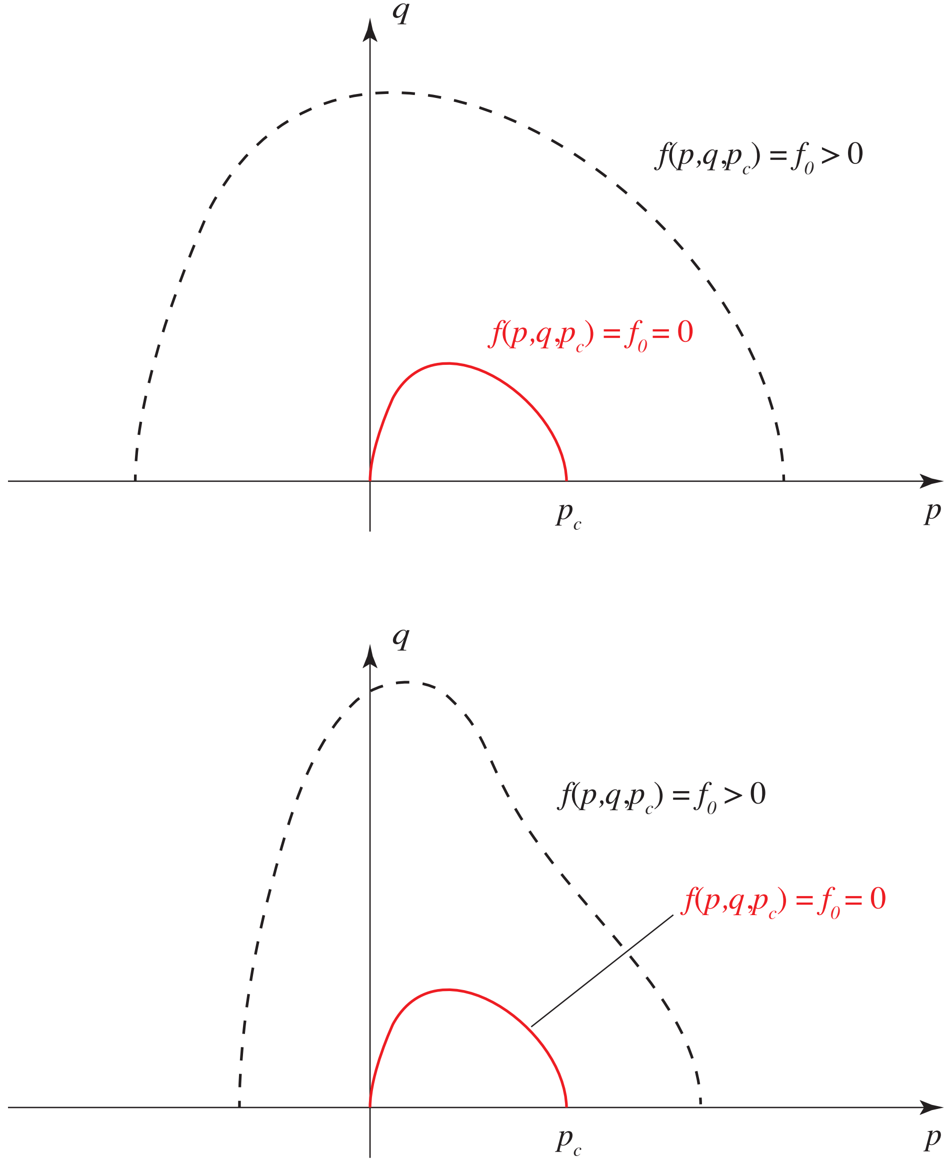
are convex for any $f_0 \in \mathbb{R}$. This relationship needs to be satisfied for any p_c

2. The model: convexity (ii)

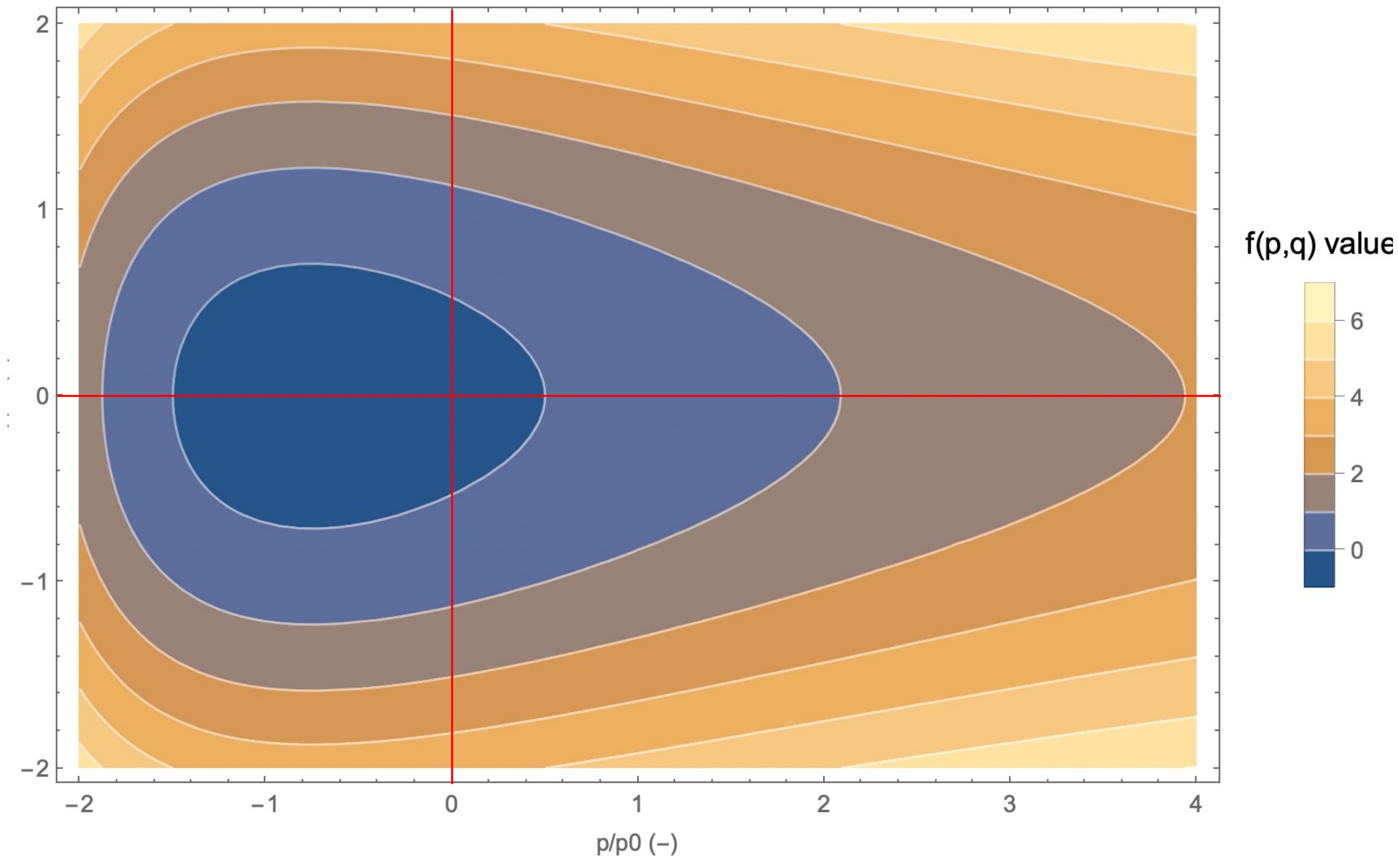
Panteghini & Lagioia (2018) describe two different types of convexity

1. **Simple convexity** indicates that only the zero level set of $f = 0$ is convex (i.e., the yield curve itself) while convexity is lost for all or some values $f = f_0 \neq 0$
2. **Full convexity** indicates that the yield function f is a quasi-convex function, so that any level set $f = f_0$ is convex

The two authors proposed the **convexification technique** to pass from simple to full convexity and to obtain also **linear homothety**



2. The model: convexity (iii)



$$M = 1, \alpha = 0.5, p_{\mathrm{m}} = p_{\mathrm{t}}$$

Here, $\boldsymbol{f(p,q)}$ **is a simply convex function**; therefore, for "high" values $f > 0$, the convexity could be lost

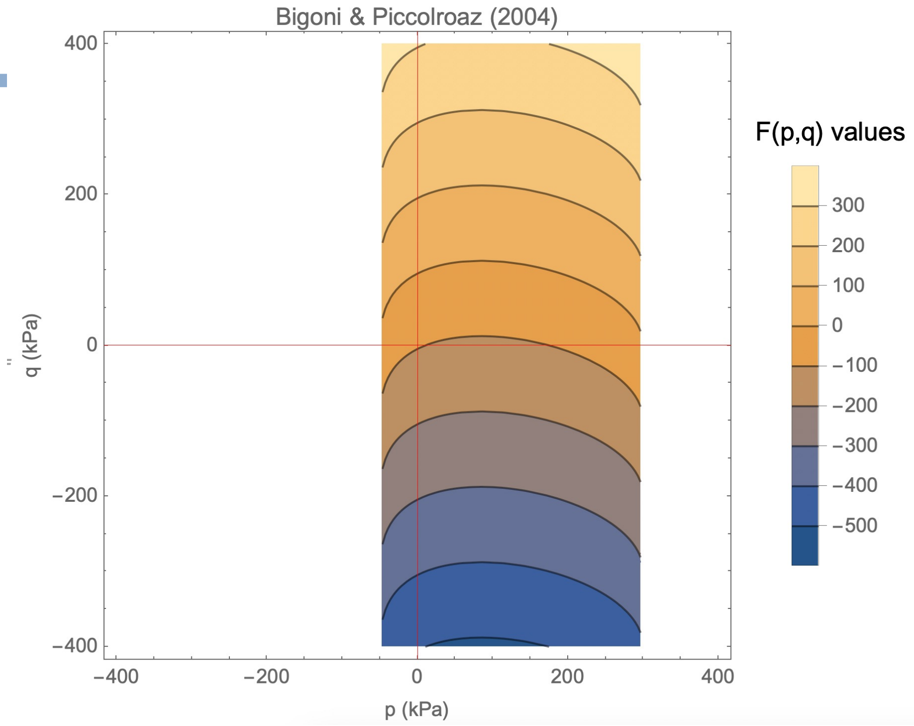
In case of Perzyna's visco-plasticity this could be a problem even if, for usual snow applications, f never reaches values higher than 2

The convexification technique is difficult to implement

An expression similar to f is used for the **visco-plastic strain potential** g together with a **non-associative flow rule**

$\boldsymbol{g}$ **is simply-convex** as well, and its definition ensures that g is null for any stress state (p, q) and the direction of the visco-plastic strains is not affected by the simple convexity

2. The model: convexity (iv)



In literature exist different surface that can change their shape but can have some problems

For instance, the **Bigoni and Piccolroaz (2004)** surface is defined only in a reduced part of the p-q plane

As a possible improvement, a fully convex yield surface could be introduced by means of the convexification process described by Panteghini & Lagioia (2017)

The fully convex surface maintains its convexity when $f > 0$