Navier–Stokes equations in Fractional Time and Multi-Fractional Space

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EGU General Assembly 2023

Vienna, Austria & Online | 23–28 April 2023

nature › scientific reports › articles › article

Article | Open Access | Published: 11 November 2022

Generalizations of incompressible and compressible Navier–Stokes equations to fractional time and multi-fractional space

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Scientific Reports **12**, Article number: 19337 (2022) | Cite this article

DOI: 10.1038/s41598-022-20911-3

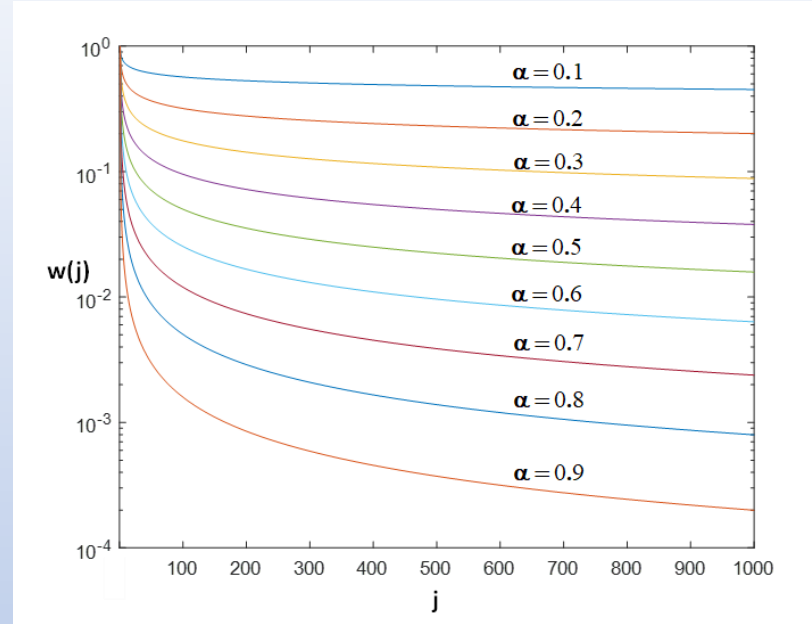
Introduction to Fractional Differentiation

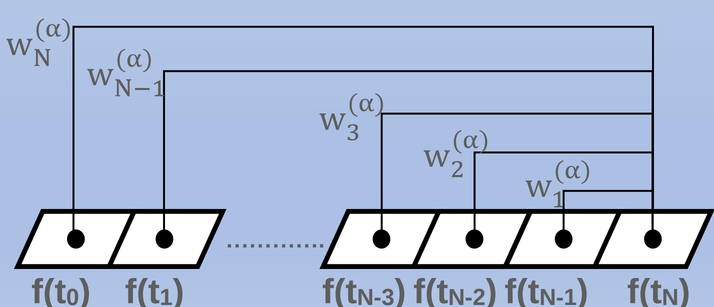
- Real numbers vs integers & fractional vs integer order derivative
- Dates back to Leibniz (1695)
- can describe memory of materials and processes
- Applications in physics, finance, bioengineering, continuum mechanics, etc.

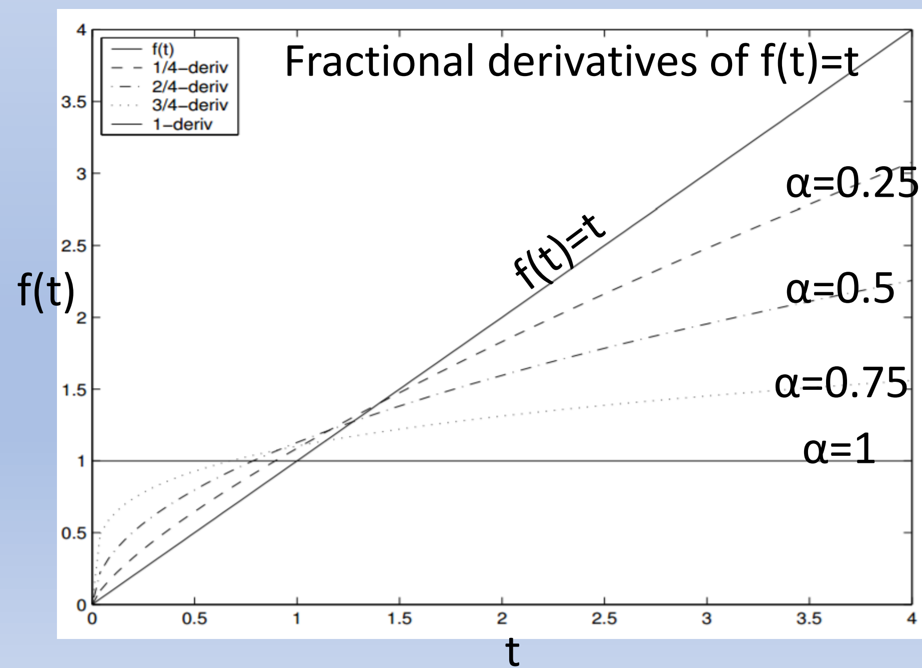
Fractional derivative in Caputo sense:

$$ {}_{t_a}^{C}D_t^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_a}^{t} \frac{u'(\tau)}{(t-\tau)^{\alpha}} d\tau $$

$$ \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt $$







- **This study develops the governing equations of unsteady multi-dimensional incompressible and compressible flow in fractional time and multi-fractional space.**

- **When their fractional powers in time and in multi-fractional space are specified to unit integer values, the developed fractional equations of continuity and momentum for incompressible and compressible fluid flow reduce to the classical Navier-Stokes equations.**

- **As such, these fractional governing equations for fluid flow may be interpreted as generalizations of the classical Navier-Stokes equations.**

Continuity Equation

$$\frac{\partial^{\alpha}\rho(\bar{x},t)}{(\partial t)^{\alpha}} = -\sum_{i=1}^{3} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \frac{\Gamma(2-\beta_i)}{{x_i}^{1-\beta_i}} \left(\frac{\partial}{\partial x_i}\right)^{\beta_i} \left(\rho(\bar{x},t)u_i(\bar{x},t)\right)$$

$$\frac{M/L^3}{T^{\alpha}} = \frac{T^{1-\alpha}}{L^{1-\beta_i}} \frac{1}{L^{\beta_i}} \frac{M}{L^3} \frac{L}{T} = \frac{M/L^3}{T^{\alpha}}$$

Conventional form is reached when α and $\beta_i \rightarrow 1$ (i = 1, 2, 3)

$$\frac{\partial\rho(\bar{x},t)}{\partial t} = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\rho(\bar{x},t)u_i(\bar{x},t)\right)$$

Continuity Equation for incompressible fluid flow

$$\sum_{i=1}^{3} \frac{\Gamma(2-\beta_i)}{{x_i}^{1-\beta_i}} \left(\frac{\partial}{\partial x_i}\right)^{\beta_i} \left(u_i(\bar{x},t)\right) = 0$$

Conventional form is reached when α and $\beta_i \rightarrow 1$ (i = 1, 2, 3)

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(u_i(\bar{x},t)\right) = 0$$

Momentum equations for compressible fluid flow under Stokes viscosity law

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\frac{\partial^{\alpha}\rho(\overline{x},t)u_i(\overline{x};t)}{(\partial t)^{\alpha}} = -\sum_{j=1}^{3}\frac{\Gamma\left(2-\beta_j\right)}{x_j^{1-\beta_j}}\left(\frac{\partial}{\partial x_j}\right)^{\beta_j}\left(\rho(\overline{x},t)u_i(\overline{x};t)u_j(\overline{x};t)\right)$$

$$+\rho(\overline{x},t)g_i(\overline{x},t)-\frac{\Gamma(2-\beta_i)}{x_i^{1-\beta_i}}\left(\frac{\partial}{\partial x_i}\right)^{\beta_i}(P(\overline{x},t))$$

$$+\sum_{j=1}^{3}\frac{\Gamma\left(2-\beta_j\right)}{x_j^{1-\beta_j}}\left(\frac{\partial}{\partial x_j}\right)^{\beta_j}\left[\mu\frac{\Gamma\left(2-\beta_j\right)}{x_j^{1-\beta_j}}\frac{\partial^{\beta_j}u_i}{(\partial x_j)^{\beta_j}}+\mu\frac{\Gamma(2-\beta_i)}{x_i^{1-\beta_i}}\frac{\partial^{\beta_i}u_j}{(\partial x_i)^{\beta_i}}\right]$$

$$-\frac{\Gamma(2-\beta_i)}{x_i^{1-\beta_i}}\left(\frac{\partial}{\partial x_i}\right)^{\beta_i}\left[\frac{2}{3}\mu\sum_{k=1}^{3}\frac{\Gamma(2-\beta_k)}{x_k^{1-\beta_k}}\frac{\partial^{\beta_k}u_k}{(\partial x_k)^{\beta_k}}\right], i=1,2,3$$

Conventional form is reached when α and $\beta_i \rightarrow 1$ (i = 1, 2, 3)

$$\rho(\overline{x},t)\frac{\partial u_i(\overline{x};t)}{\partial t} = -\rho(\overline{x},t)\sum_{j=1}^{3} u_j(\overline{x};t)\frac{\partial}{\partial x_j}(u_i(\overline{x};t)) + \rho(\overline{x},t)g_i(\overline{x},t) - \frac{\partial}{\partial x_i}(P(\overline{x},t)) + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left[\mu\frac{\partial u_i}{\partial x_j} + \mu\frac{\partial u_j}{\partial x_i}\right] - \frac{\partial}{\partial x_i}\left[\frac{2}{3}\mu\sum_{k=1}^{3}\frac{\partial u_k}{\partial x_k}\right], i = 1,2,3$$

Momentum equations for incompressible fluid flow under Stokes viscosity law

$$\rho\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\frac{\partial^{\alpha}u_i(\overline{x};t)}{(\partial t)^{\alpha}} = -\rho\sum_{j=1}^{3}\frac{\Gamma(2-\beta_j)}{x_j^{1-\beta_j}}\left(\frac{\partial}{\partial x_j}\right)^{\beta_j}(u_i(\overline{x};t)u_j(\overline{x};t))$$
$$+\rho g_i(\overline{x},t)-\frac{\Gamma(2-\beta_i)}{x_i^{1-\beta_i}}\left(\frac{\partial}{\partial x_i}\right)^{\beta_i}(P(\overline{x},t))$$
$$+\mu\sum_{j=1}^{3}\frac{\Gamma(2-\beta_j)}{x_j^{1-\beta_j}}\left(\frac{\partial}{\partial x_j}\right)^{\beta_j}\left(\frac{\Gamma(2-\beta_j)}{x_j^{1-\beta_j}}\frac{\partial^{\beta_j}u_i}{(\partial x_j)^{\beta_j}}\right), \mathrm{i}=1,2,3$$

Conventional form is reached when α and $\beta_i \rightarrow 1$ (i = 1, 2, 3)

$$\rho\frac{\partial u_i(\overline{x};t)}{\partial t} = -\rho\sum_{j=1}^{3}u_j(\overline{x};t)\frac{\partial}{\partial x_j}(u_i(\overline{x};t))+\rho g_i(\overline{x},t)-\frac{\partial}{\partial x_i}(P(\overline{x},t))+\mu\sum_{j=1}^{3}\frac{\partial^2 u_i}{\partial x_j^2},\quad \mathrm{i}=1,2,3$$

Numerical Application:

The first Stokes Problem (i.e., flow due to a wall suddenly set into motion)

A fluid with constant density and viscosity is bounded by a solid wall (at $x_2 = 0$), which is set in motion in positive x_1 direction at t=0 with a constant velocity U_0.

$$\rho \frac{\partial u_1(x_2, t)}{\partial t} = \mu \frac{\partial^2 u_1(x_2, t)}{\partial {x_2}^2}$$

the initial condition

$$u_1(x_2, t = 0) = 0;$$

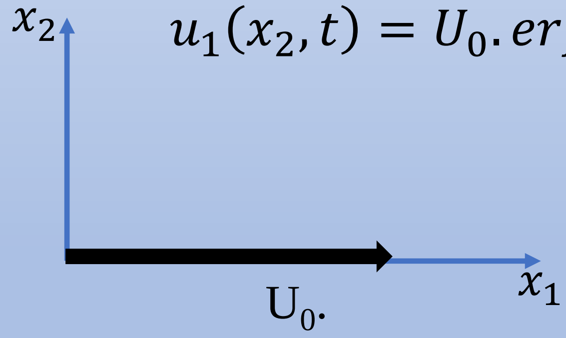
boundary conditions:

$$u_1(x_2 = 0, t \geq 0) = U_0;$$

$$u_1(x_2 = \infty, t \geq 0) = 0.$$

analytical solution (Bird et al.[55])

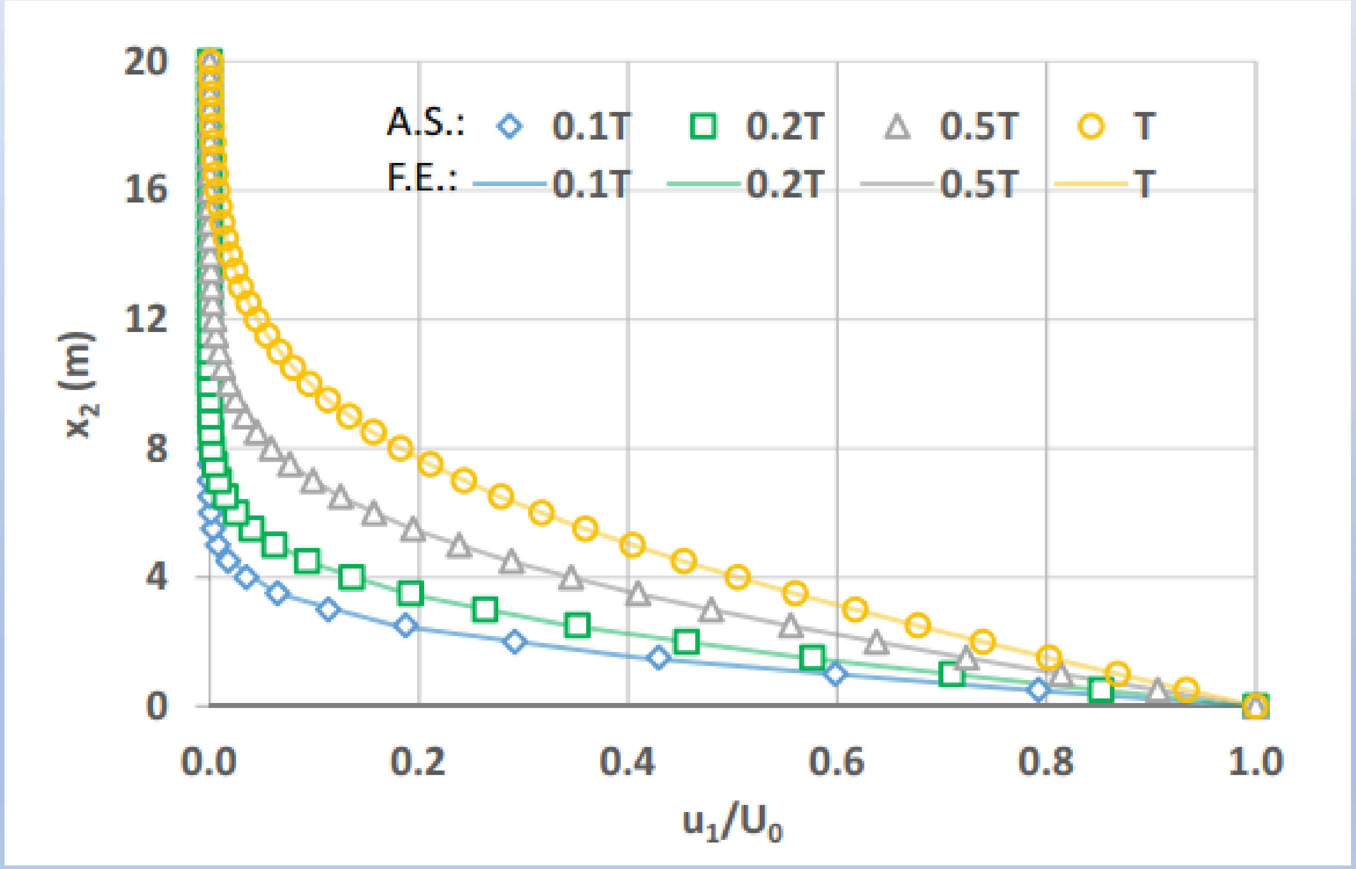
$$u_1(x_2, t) = U_0 . erfc\left(\frac{x_2}{\sqrt{4\nu t}}\right)$$



$\nu = 0.001$ Pa.s, T = 5 hours, and U_0=1 m/s.

The fractional form of this problem can be written as

$$\frac{\partial^{\alpha} u_1(x_2,t)}{(\partial t)^{\alpha}} = \nu \frac{t^{1-\alpha}}{{x_2}^{1-\beta}} \frac{\Gamma(2-\alpha)}{\Gamma(2-\beta)} \left(\frac{\partial}{\partial x_2}\right)^{\beta} \frac{\Gamma(2-\beta)}{{x_2}^{1-\beta}} \frac{\partial^{\beta} u_1}{(\partial x_2)^{\beta}}$$



Velocity profiles when space and time fractional derivative powers are 1.

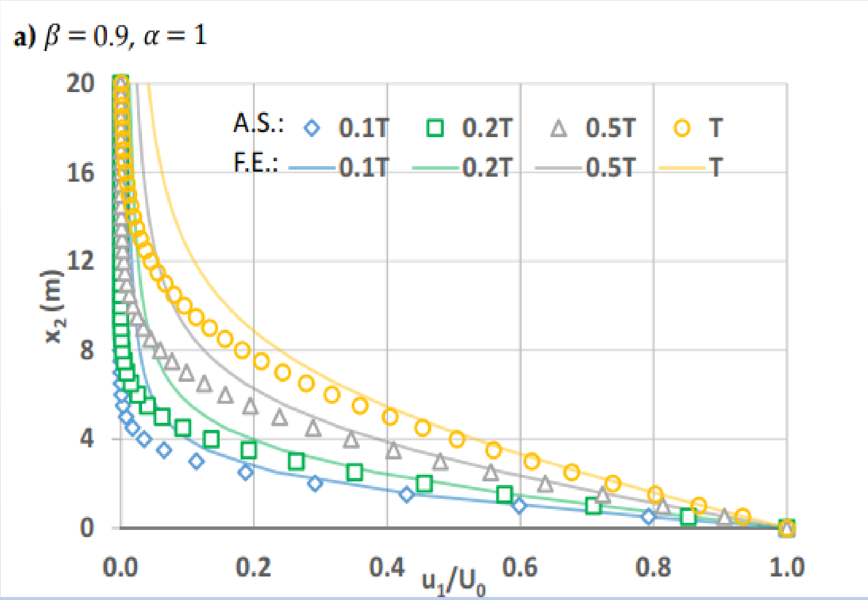
Non-Fickian flow processes (Zaslavsky[30])

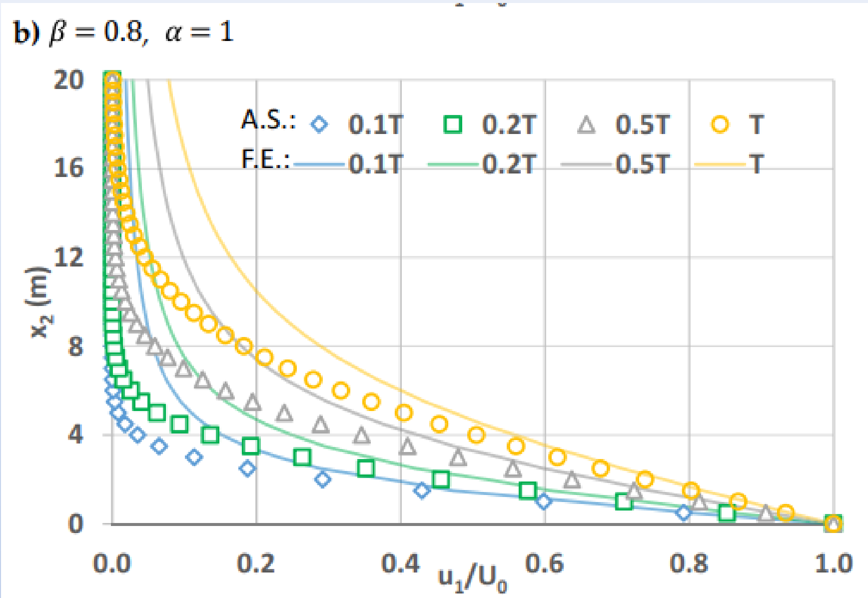
Sub-diffusive (slow) flow: $\mu<1$
Normal diffusive flow: $\mu=1$
Super-diffusive (fast) flow: $\mu>1$

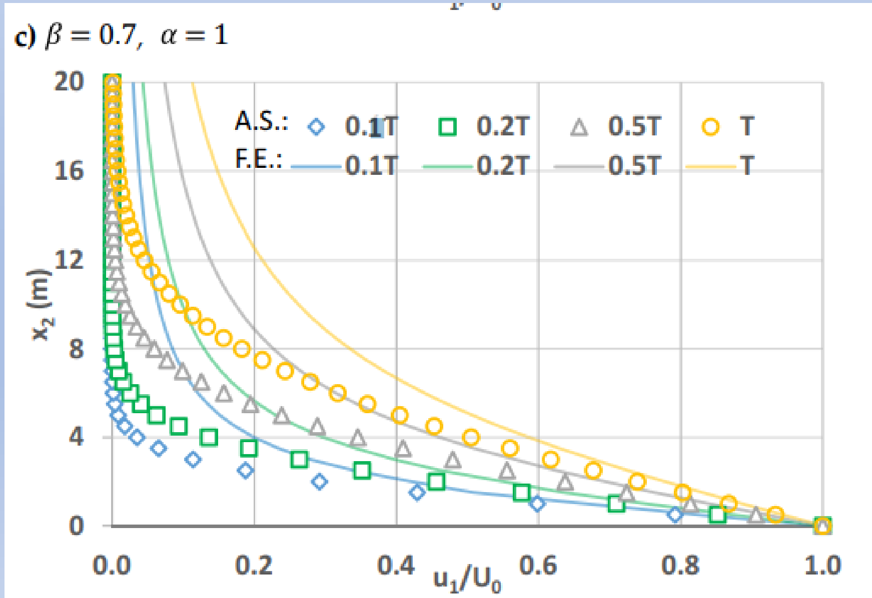
where transport exponent $\mu= \alpha/\beta$
(ratio of time to space fractional powers)

Time fractional derivative, α=1

β <1 and μ= α/β >1 (super-diffusion)

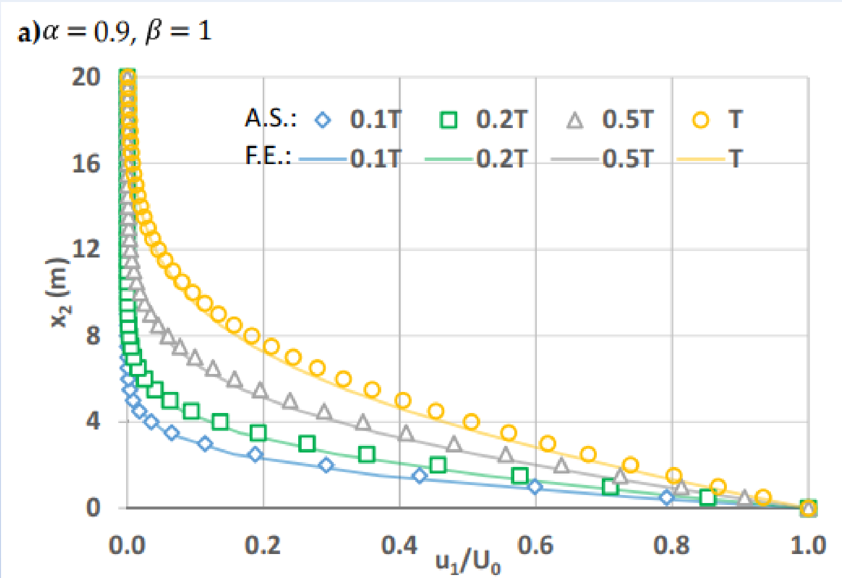


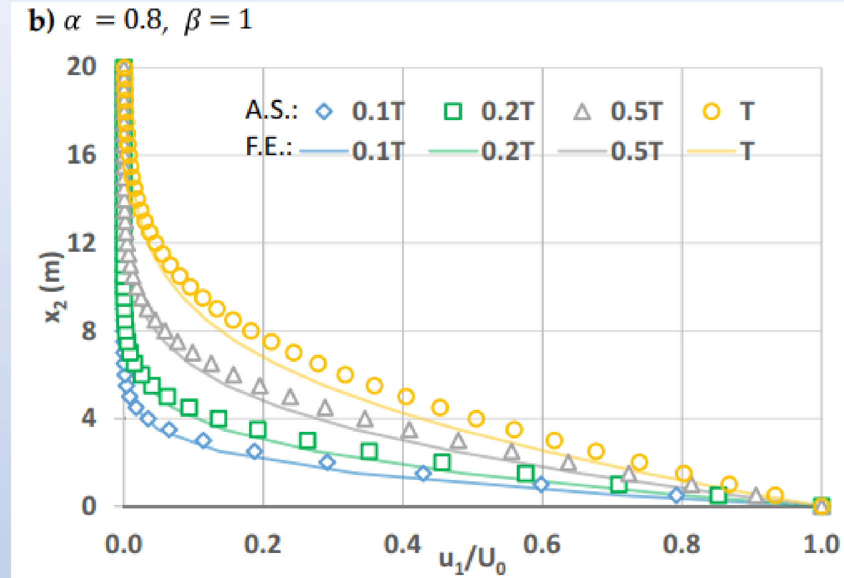


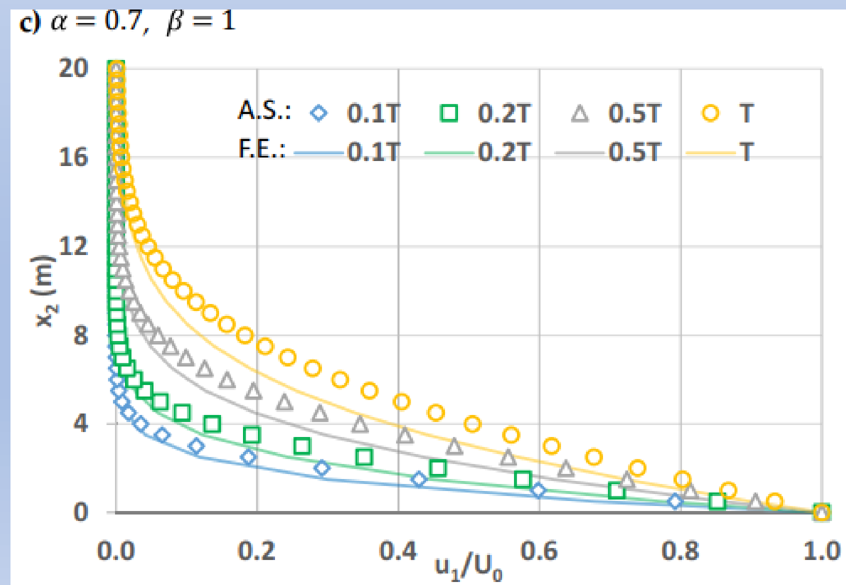


Space fractional derivative, β =1

α <1 and μ= α/β <1 (sub-diffusion)

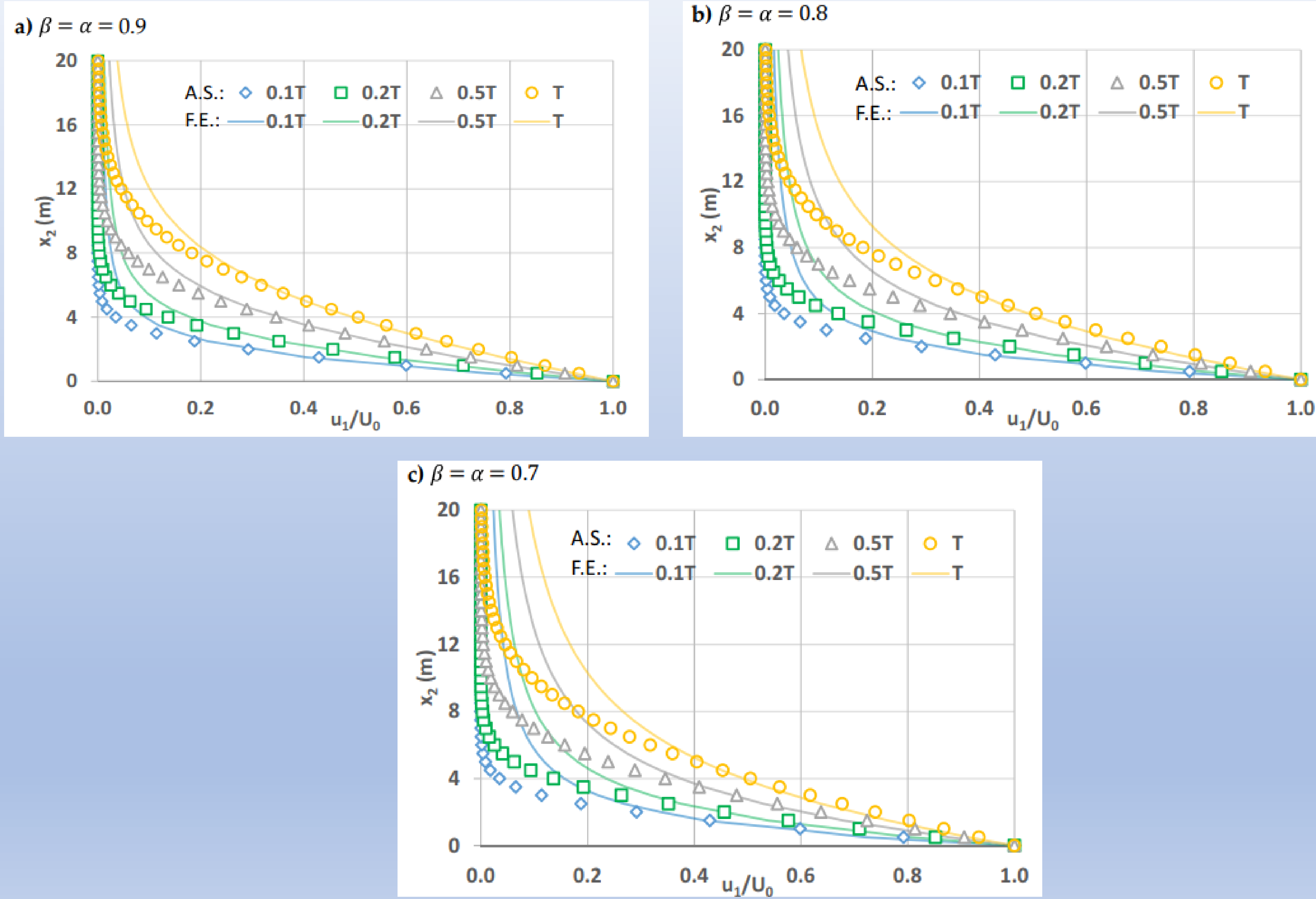






α = β <1

μ= α/β =1 (normal-diffusion)



Conclusion

- **Proposed fractional governing equations for fluid flow are generalizations of the classical Navier-Stokes equations.**

- **The derived governing equations of fluid flow in fractional differentiation framework are nonlocal in time and space. Therefore, they can quantify the effects of initial and boundary conditions better than the classical Navier-Stokes equations.**

- **Proposed fractional equations of fluid flow have the potential to accommodate both the sub-diffusive and the super-diffusive flow conditions.**