Dependence of Early Warning Signals on Time Scale Separation

CriticalEarth

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Introduction

The system under study is a two dimensional stochastic FitzHugh-Nagumo type system with a time scale separation.

$$dx = (a_1 x - a_3 x^3 + by)dt + \sigma dW_t = f(x,y)dt + \sigma dW_t \quad (1)$$
$$dy = \varepsilon(\beta y - x + c)dt = \varepsilon g(x,y)dt \quad (2)$$

The x variable is the fast dynamics and are modified with a noise term. The y variable is the slow dynamics, with the parameter $\varepsilon \leq 1$ determining the separation of time scales. In theory, this system only experiences bifurcations for infinite time scale separation ($\varepsilon \to 0$), but intermediary values also exhibit rich dynamics.

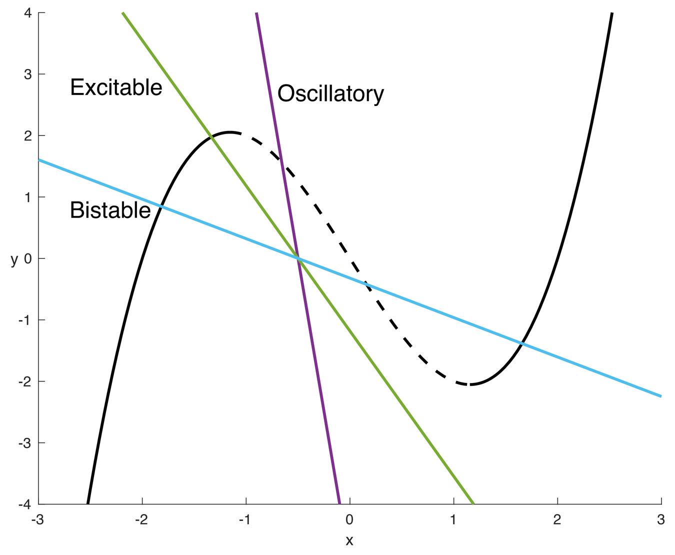

Figure (1) Nullclines of the system (1), (2). Text describes the dynamical regimes of the system for different values of parameter β

The x variable has a cubic nullcline with two folds at the local extrema. The orientation of the linear nullcline (Figure 1) determines which of the three dynamical regimes the system is in. For the bistable regime, the system admits two stable and one unstable equilibrium points. Stochastic forcing can cause the system to switch between the two stable states. In the excitable regime, there is just one stable equilibrium. However, noise can cause an excursion to the other branch, before returning to the equilibrium. Finally, the oscillatory regime has an unstable fixed point, and the system oscillates along and between the two stable branches, Of course, large enough noise can cause a transition in these latter two regimes as well, by causing the trajectory to cross the unstable manifold of the x nullcline. This study focuses on the oscillatory regime.

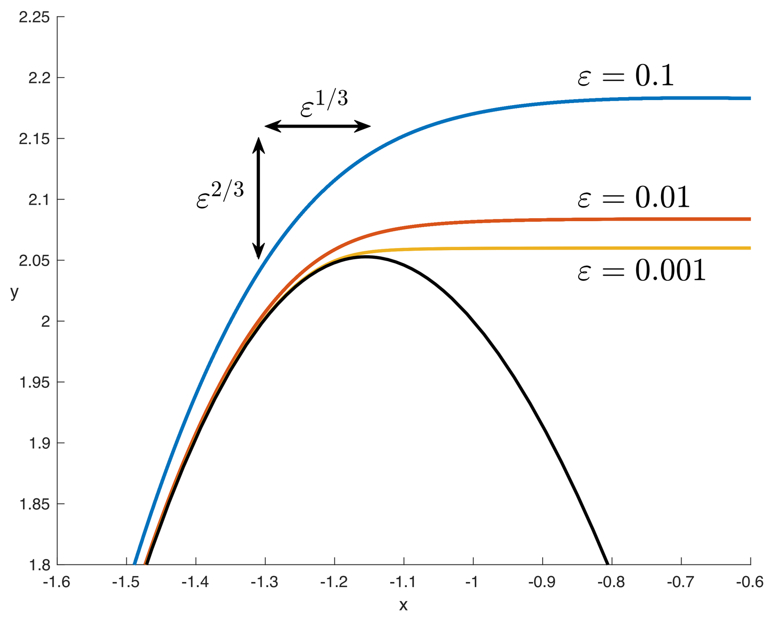

Figure (2) Deterministic trajectories for different values of time scale separation ε. Trajectories deviate from the critical manifold on the order of powers of ε near the fold bifurcation.

Figure 2 shows how the deterministic dynamics behave for different values of ε. Trajectories track the slow manifold, which is order ε from the stable branches of the critical manifold $f(x,y) = 0$. Nearer to the bifurcation point, there is a deviation from the critical manifold on the order of $\varepsilon^{2/3}$ in the fast variable and $\varepsilon^{1/3}$ in the slow variable. Thus for larger values of ε, the system tips much later than the bifurcation point is reached.

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Early Warning Signals

In stochastic time series, early warning signals (EWS) can detect critical transitions such as bifurcations in the dynamics by measuring statistical indicators such as variance and autocorrelation. As the system approaches a fold bifurcation, the restoring rate to the equilibrium weakens as the eigenvalues of the Jacobian approach the imaginary axis. This causes the stochastic trajectory to meander, resulting in an increase in variance and autocorrelation in a phenomenon known as critical slowing down. While there are other indicators that can be used as EWS to detect critical transition, these two are the most straightforward and the focus of this study.

We look the data in a sliding window approaching the critical transition. This window must be long enough that we can recover acceptable statistics, but not too long as to lose quasi-stationarity of the distribution of the signal in this window. Since the variance and autocorrelation approach and asymptote at the bifurcation, we may only look at a certain time before it.

Questions:

- do we see an increase in variance and autocorrelation that will tell us that a transition will occur?
- can we predict when this transition will occur?

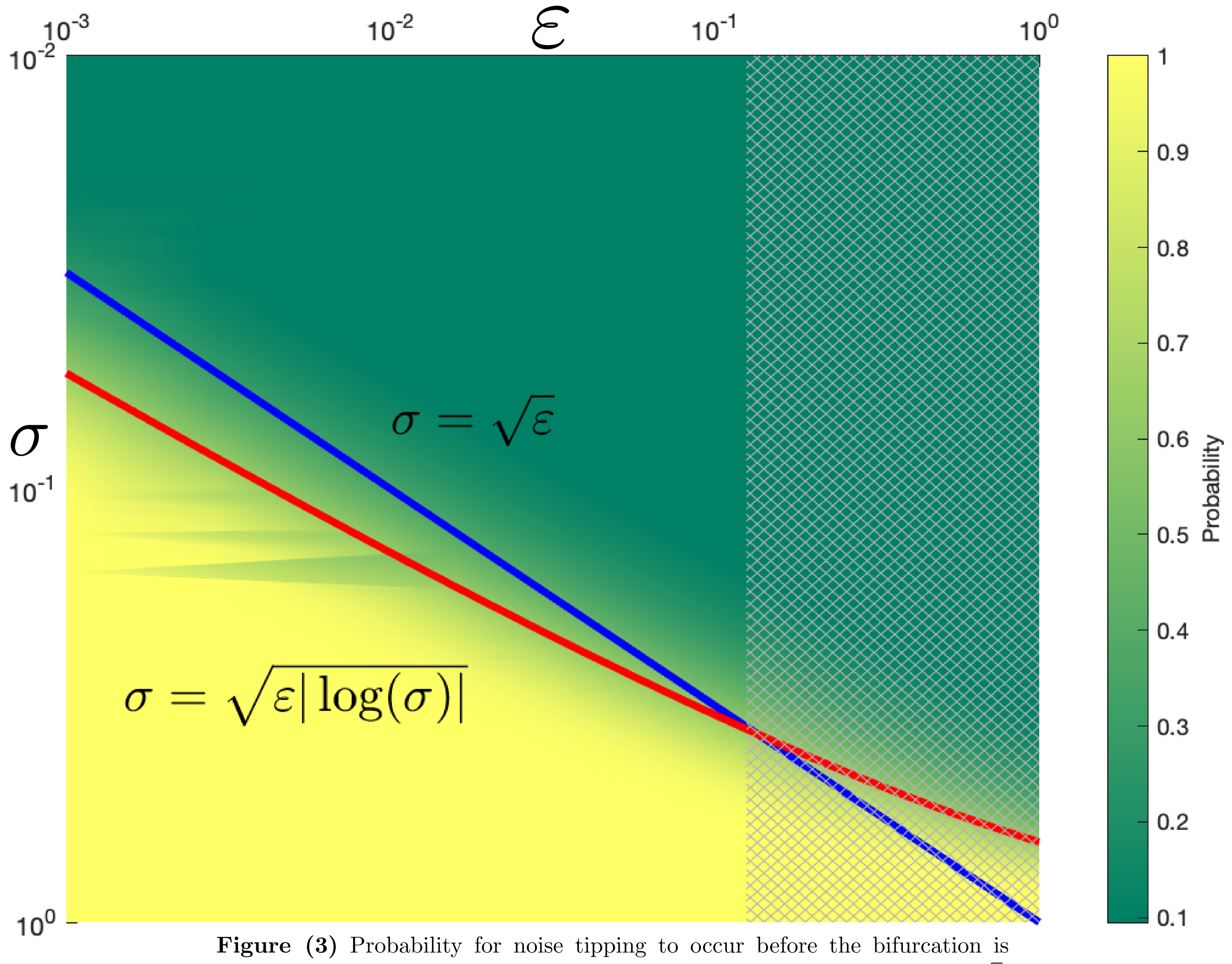

Figure (3) Probability for noise tipping to occur before the bifurcation is reached depending on ε and σ. The blue line indicates the equality $\sigma = \sqrt{\varepsilon}$, which is a lower bound for noise tipping to be probable The red line indicates the equality $\sigma = \sqrt{\varepsilon|\log(\sigma)|}$, which is a lower bound for noise tipping to be likely. The grey hatches show an approximate region where EWS cannot be detected.

Figure 3 shows the probability that a noise-induced transition will occur before the bifurcation point is reached for a given value of ε and σ. For larger values of σ, the probability is close to 1. We can delineate this region by two curves: The blue curve $\sigma = \sqrt{\varepsilon}$ splits the parameter space into the strong noise ($\sigma > \sqrt{\varepsilon}$) and the weak noise ($\sigma < \sqrt{\varepsilon}$) regimes. In the weak noise regime, the system is very unlikely to tip before the bifurcation, and we would expect to be able to see EWS and predict the time of tipping (see Figure 4). The red curve shows $\sigma = \sqrt{\varepsilon|\log(\sigma)|}$. For values of σ larger than this, the probability of tipping before the fold is close to 1. In this regime, we can still see an increase of variance and autocorrelation, but the system is likely to tip due to noise and the time of tipping is unpredictable (see Figure 5).

Additionally, the grey hatches ($\varepsilon > 0.125$) show an approximate region within which EWS are not visible (see Figure 6). For $\varepsilon \sim 1$, the system is too far from the fold point, and does not tip. Thus in this region we do not see EWS, even for very low noise.

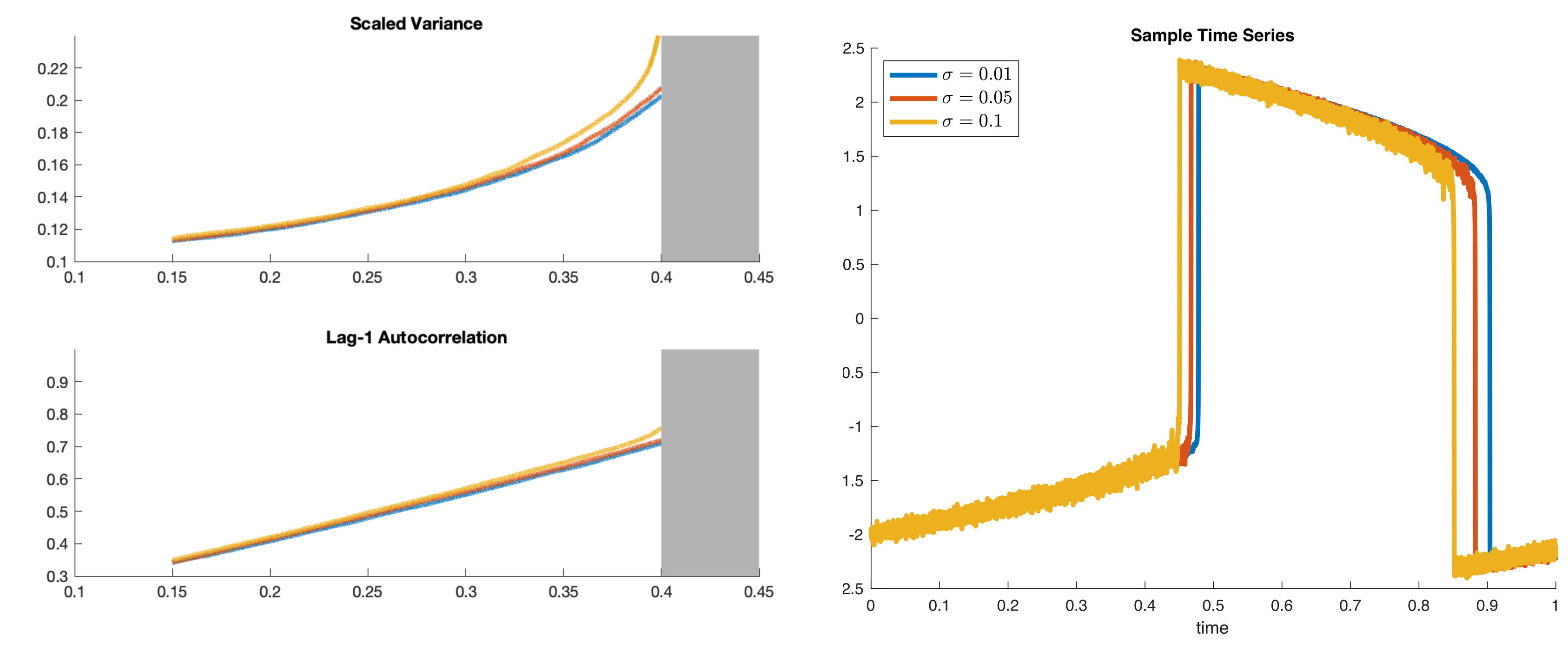

Figure (4) Sample time series and ensemble average EWS for $\varepsilon = 0.002$ and three different noise strengths

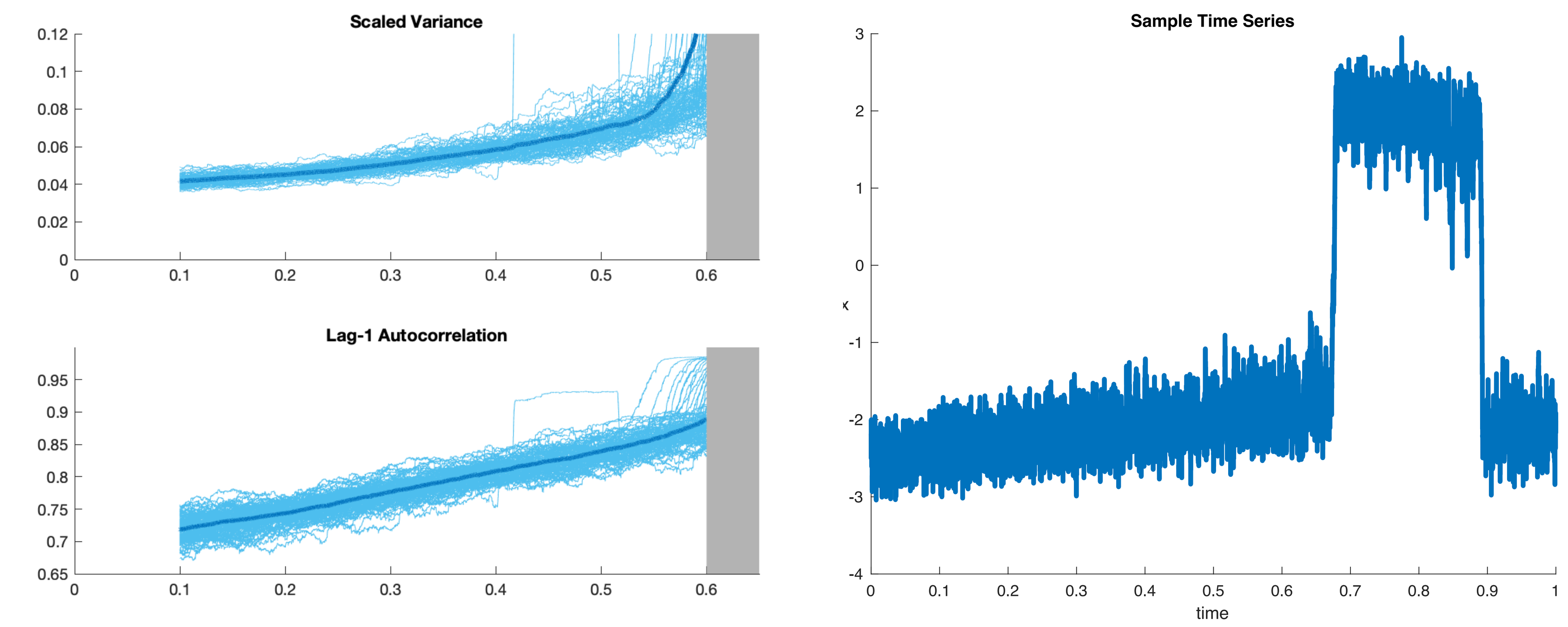

Figure (5) Sample time series and ensemble average EWS for $\varepsilon = 0.01$ and $\sigma = 1$

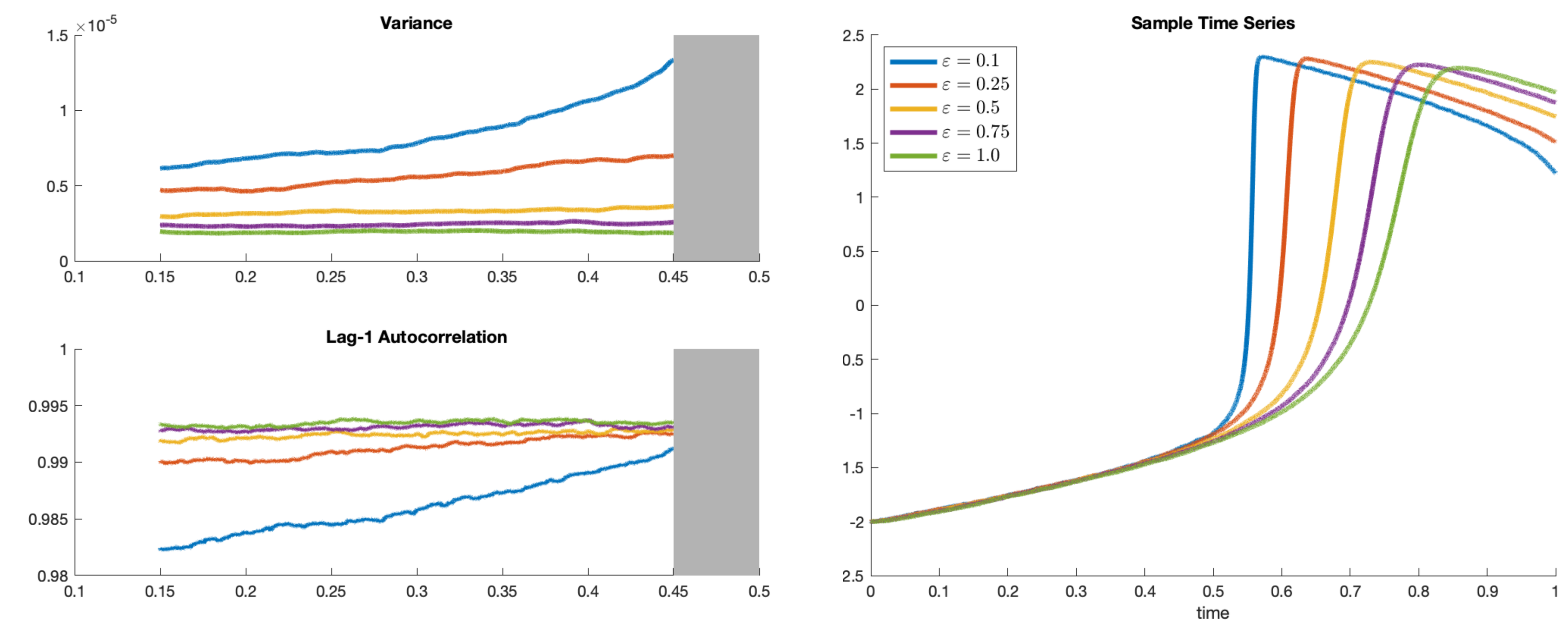

Figure (6) Sample time series and ensemble average EWS for ε near 1 and $\sigma = 0.01$

Conclusion

The parameter space of time scale separation ε and noise strength σ can be split into three distinct regions:

1. large ε: cannot predict if a transition will occur
2. small ε, strong noise: can predict that a transition will occur, but not when
3. small ε, weak noise: can predict that a transition will occur and when