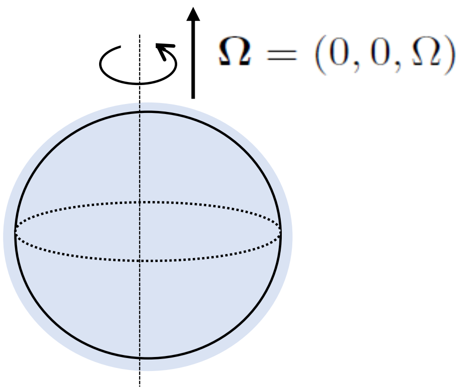
Rossby wave nonlinear interactions and large-scale zonal flow formation in two-dimensional turbulence on a rotating sphere

Apologies: probably should have been presented in another session.

EGU24-15282

EGU Meeting 2024
19th April, 2024
at Austria Center Vienna



Y. Hagimori[1], K. Obuse[1], and M. Yamada[2]
1: Okayama University
2: Kyoto University

Menu of the day

- **2D Navier-Stokes turbulence on a rotating sphere**
  - equations used in this talk
  - zonal flow structure in 2D turbulence on a rotating system

- **Rossby waves**
  - flow dynamics and Rossby waves
  - resonant interaction of Rossby waves

- **Nonlinear interactions of Rossby waves and zonal flow formation**
  - Near-Resonant interactions and zonal flow formation
  - Non-Local energy transfer and zonal flow formation

- Summary and Discussion

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2D turbulence on a rotating sphere

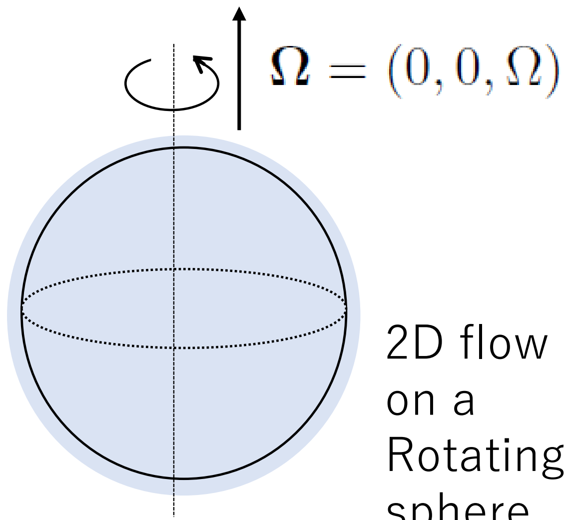
The system I am mainly interested in ⋯

➢ 2D turbulence on a rotating sphere

Interested in ...：

- the characteristics of the solution of Navier-Stokes equations
- inhomogeneous and anisotropic structure (zonal flow) formation
- in relation to planetary atmospheres and oceans
(foundation of more complicated and realistic mathematical models)
- in relation to plasma physics (Hasegawa-Mima equation)

(Sphere→beta-plane→H-M equation)



2D flow
on a
Rotating
sphere

2D Navier-Stokes equations and Vorticity equation on a rotating sphere

2-dimentional Navier-Stokes equations on a rotating sphere:

$$\frac{\partial u}{\partial t} + \frac{u}{\cos\phi}\frac{\partial u}{\partial \lambda} + v\frac{\partial u}{\partial \phi} - uv\tan\phi - 2\Omega\sin\phi v = -\frac{1}{\rho\cos\phi}\frac{\partial p}{\partial \lambda} + D_\lambda + F_\lambda,$$

$$\frac{\partial v}{\partial t} + \frac{u}{\cos\phi}\frac{\partial v}{\partial \lambda} + v\frac{\partial v}{\partial \phi} - v^2\tan\phi - 2\Omega\sin\phi u = -\frac{1}{\rho}\frac{\partial p}{\partial \phi} + D_\phi + F_\phi,$$

Advection(nonlinear)

rotation effect

λ : longitude,
ϕ: latitude

u: lon-velocity,
v: lat-velocity

$+$

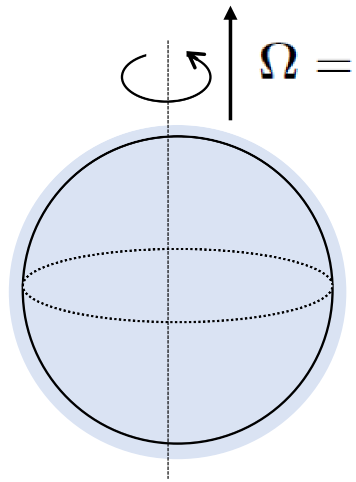
Continuity equation (incompressible flow):

$$\frac{1}{\cos\phi}\frac{\partial u}{\partial \lambda} + \frac{1}{\cos\phi}\frac{\partial (v\cos\phi)}{\partial \phi} = 0.$$

Introduce stream function ψ

Vorticity equation:

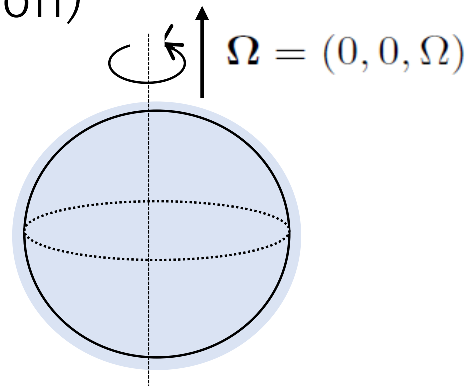
$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) + 2\Omega\frac{\partial \psi}{\partial \lambda} = \nu\left(\nabla^2 + \ 2\ \right) + F$$



2D Navier-Stokes equations and Vorticity equation on a rotating sphere

- Non-dimentionalised vorticity equation on a rotating sphere

(Navier-stokes equations + continuity equation → vorticity equation)



$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) + 2\Omega \frac{\partial \psi}{\partial \lambda} = \mathrm{D_{issipation}} + \mathrm{F_{orcing}}$$

advection (nonlinear) — $J(\psi, \zeta)$; rotation — $2\Omega \frac{\partial \psi}{\partial \lambda}$

(λ, μ) : longitude, sin(latitude), Ω: rotation rate of the sphere

$\psi(\lambda, \mu, t)$: stream function, $\zeta \equiv \nabla^2 \psi$: vorticity,

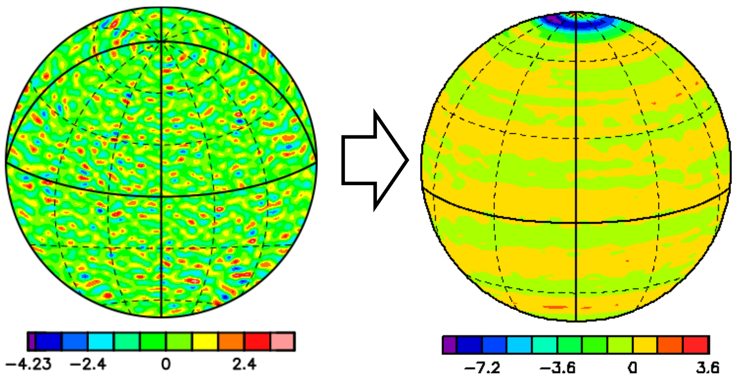
nonlinear term

$$u_{lon} = -\sqrt{1-\mu^2}\frac{\partial \psi}{\partial \mu}, \quad u_{lat} = \frac{1}{\sqrt{1-\mu^2}}\frac{\partial \psi}{\partial \lambda}, \quad J(A, B) \equiv \frac{\partial A}{\partial \lambda}\frac{\partial B}{\partial \mu} - \frac{\partial A}{\partial \mu}\frac{\partial B}{\partial \lambda}$$

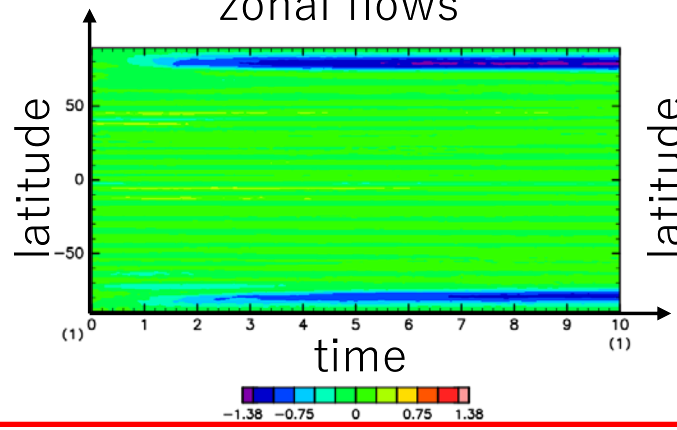
Zonal flow formation in 2D turbulence on rotating systems

- Unforced: westward circumpolar flows (Yoden and Yamada1993, Takehiro et al. 2007)

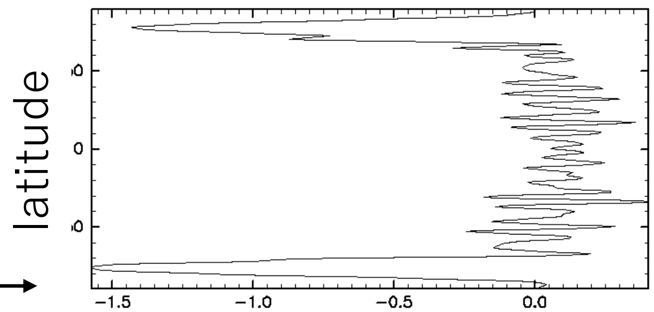
Longitudinal velocity



temporal development of zonal flows



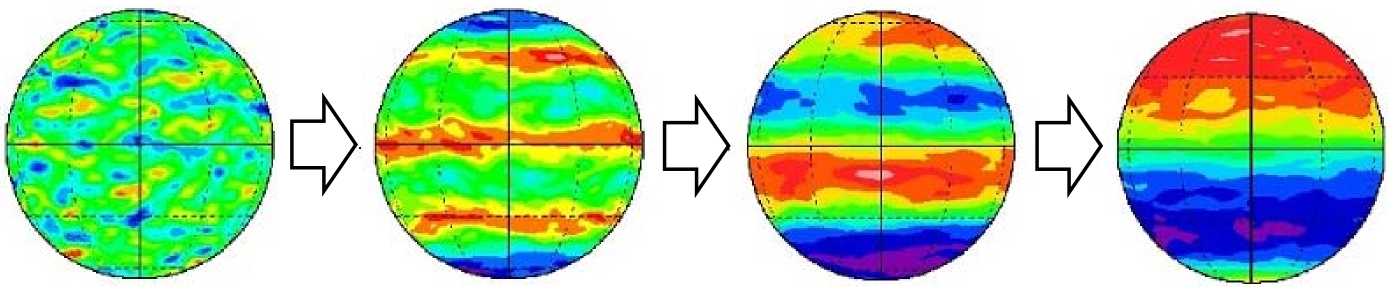
westward circumpolar jets



- Forced: multiple zonal band structure → a few large zonal flows

(Nozawa and Yoden 1997 , Obuse et al. 2010)

Longitudinal velocity



✓Zonal structure is formed and maintained for a very long time
✓ Mechanism of zonal flow formation is not yet made clear

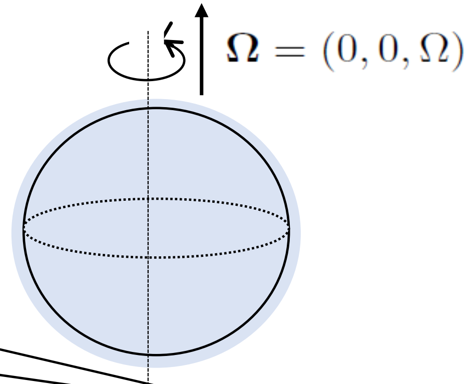
2D Navier-Stokes equations and Vorticity equation on a rotating sphere

- Non-dimentionalised vorticity equation on a rotating sphere

(**Euler** equations + continuity equation → vorticity equation)

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) + 2\Omega \frac{\partial \psi}{\partial \lambda} = \mathrm{D}_{\mathrm{issipation}} + \mathrm{F}_{\mathrm{orcing}}$$

advection (nonlinear) — rotation



(λ, μ) : longitude, sin(latitude), Ω: rotation rate of the sphere

$\psi(\lambda, \mu, t)$: stream function, $\zeta \equiv \nabla^2 \psi$: vorticity,

nonlinear term

$$u_{lon} = -\sqrt{1-\mu^2}\frac{\partial \psi}{\partial \mu}, \quad u_{lat} = \frac{1}{\sqrt{1-\mu^2}}\frac{\partial \psi}{\partial \lambda}, \quad J(A, B) \equiv \frac{\partial A}{\partial \lambda}\frac{\partial B}{\partial \mu} - \frac{\partial A}{\partial \mu}\frac{\partial B}{\partial \lambda}$$

Kato (1984), O. and Yamada (in preparation)

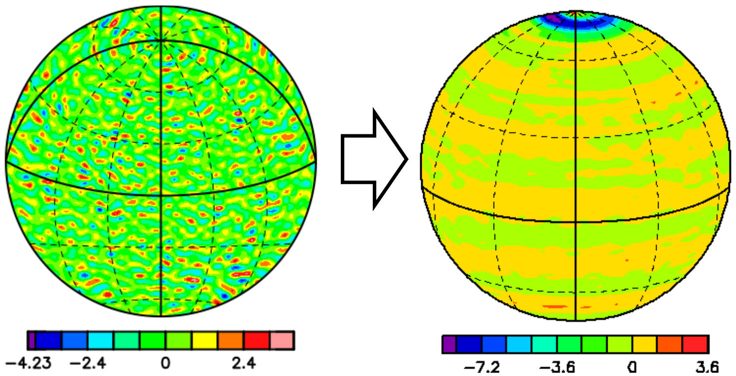
Normally Considered, but ignored in this talk

- (F:) to consider simple case
- (D:) existence of viscosity is not essential for zonal flow formation

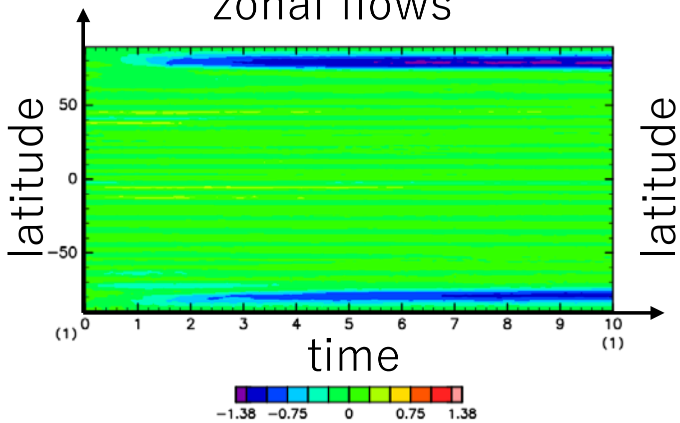
Zonal flow formation in 2D turbulence on rotating systems

- westward circumpolar zonal flows (Yoden and Yamada1993, Takehiro et al. 2007)

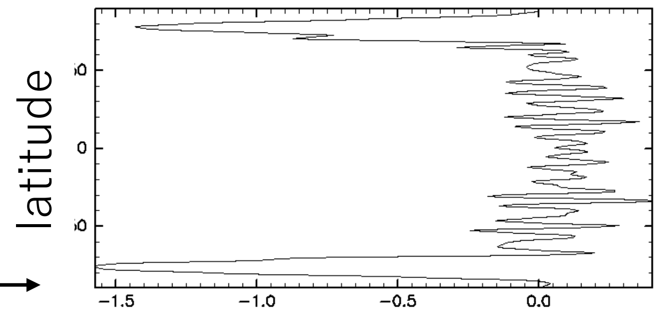
longitudinal-velocity



temporal development of zonal flows



westward circumpolar jets



Lots of study from various points of view
→Mechanism of large-scale zonal flow formation has not been made clear yet.
→Introduce our trial by using nonlinear wave interactions

Menu of the day

- **2D Navier-Stokes turbulence on a rotating sphere**
  - equations used in this talk
  - zonal flow structure in 2D turbulence on a rotating system

- **Rossby waves**
  - flow dynamics and Rossby waves
  - resonant interaction of Rossby waves

- Often used to discuss the flow dynamics in rotating system.
- We also use them today

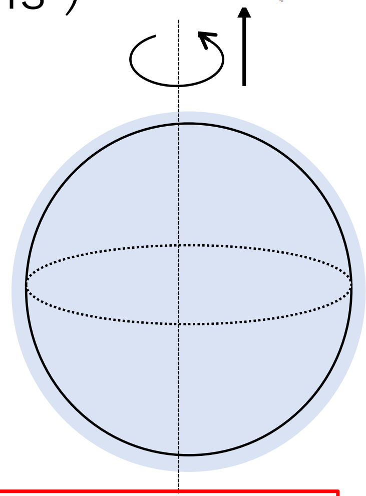
- **Nonlinear interactions of Rossby waves and zonal flow formation**
  - Near-Resonant interactions and zonal flow formation
  - Non-Local energy transfer and zonal flow formation

- Summary and Discussion

Rossby waves 2D incompressible flow on a rotating sphere

Wave solutions called **Rossby waves** (specific for rotating systems)

$\mathbf{\Omega} = (0, 0, \Omega)$



$$\begin{cases} Y_n^m(\lambda, \mu) \exp(-i\omega t), \\ \omega = \dfrac{-2m\Omega}{n(n+1)} \end{cases}$$

$$Y_n^m(\lambda, \mu) = P_n^m(\lambda, \mu) \exp(-im\lambda)$$

: spherical harmonics

Dynamics of Rossby waves determines the temporal variation of flow filed.
(Three-wave)Nonlinear interactions of Rossby waves are important!

We **investigate zonal flow formation**
from the perspective of three-Rossby-wave nonlinear interaction

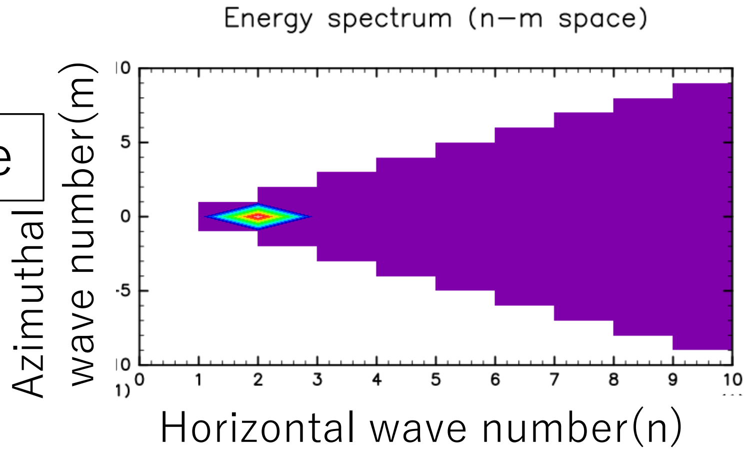
Briefly introduce

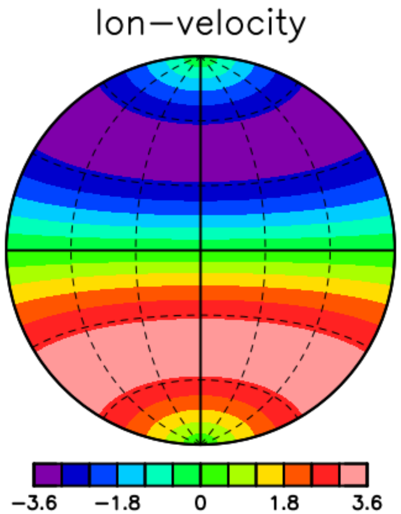
- zonal Rossby modes
- conditions for three-wave nonlinear interaction
- resonant three-wave nonlinear interaction

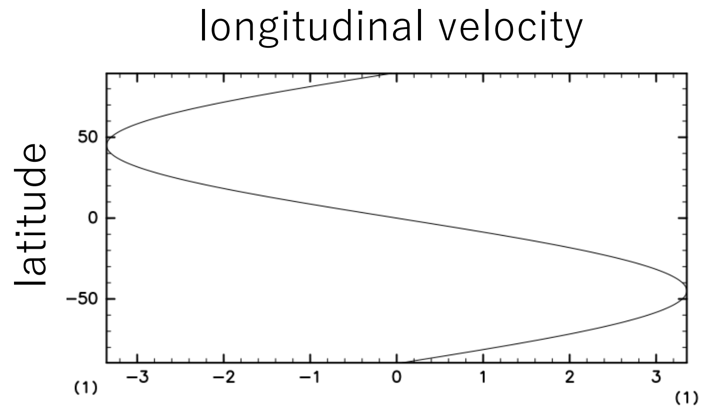
Rossby waves in charge of describing zonal flows (zonal Rossby modes)

$Y_n^0(\lambda, \mu)$ modes have zonal structures

Y_2^0 mode

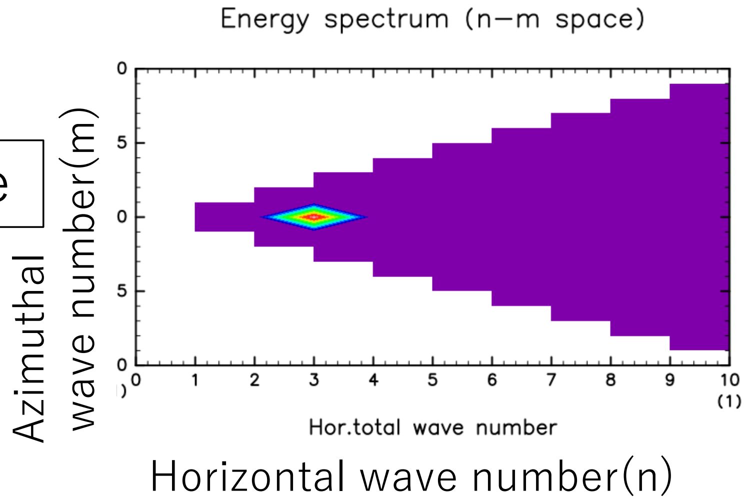


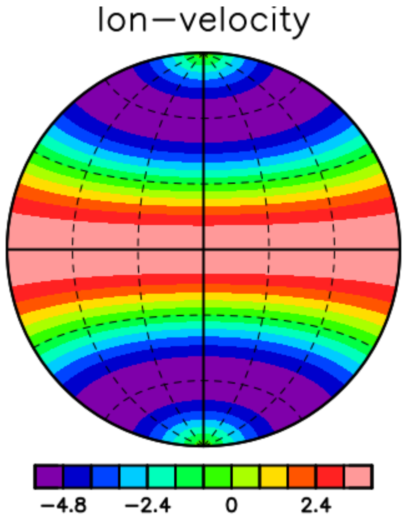


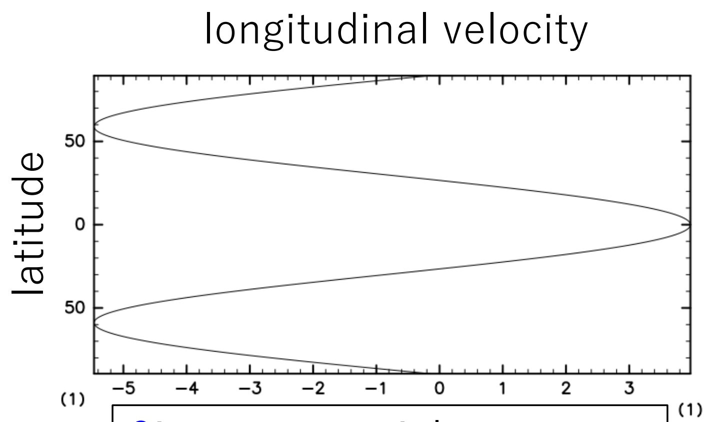


2jets, equatorial antisymmetriy

Y_3^0 mode

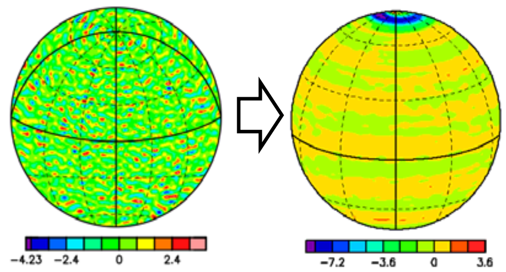






3jets, equatorial symmetry

Development of zonal Rossby modes

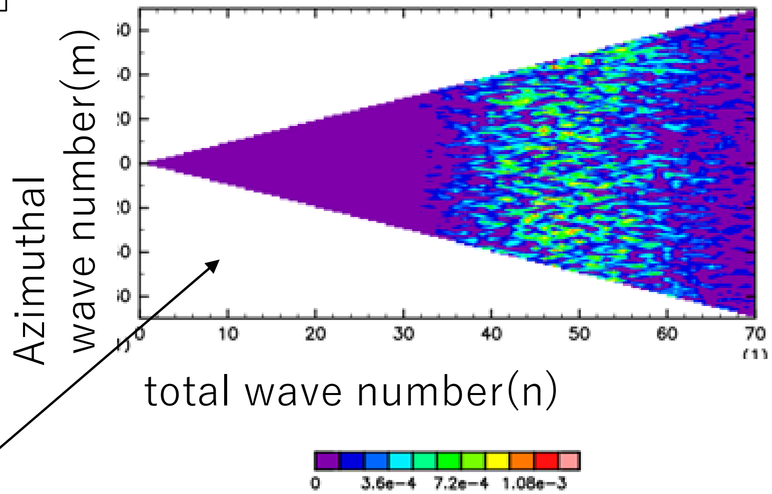


Development of zonal flow

= energy accumulation to $Y_n^0(\lambda, \mu)$ Rossby modes

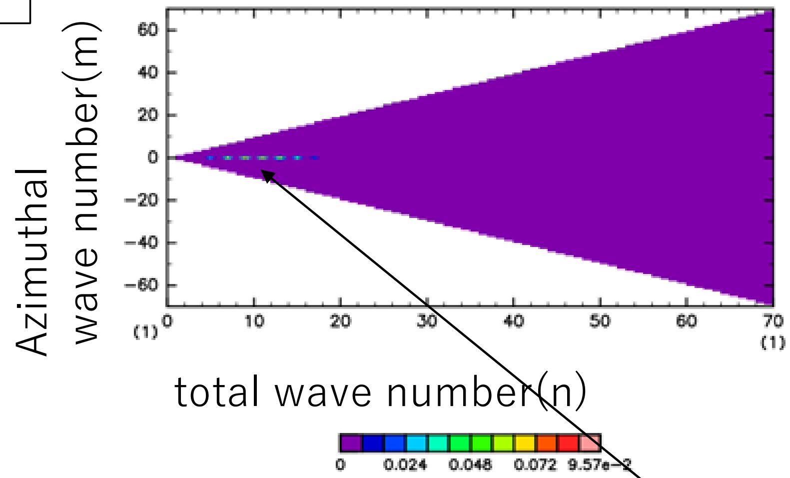
t=0

energy spectrum E_n^m(n-m space)



t=10

energy spectrum E_n^m(n-m space)



Keep in mind

Correspond to circumpolar zonal flows

Random initial condition

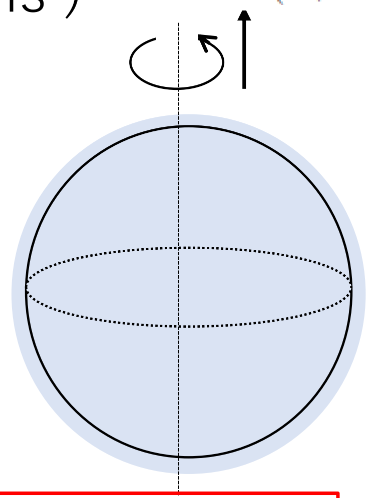
$$E(n, t=0) = \frac{A n^{\gamma/2}}{(n+n_0)^{\gamma}}, \quad n_0 = 50, \quad \gamma = 100, \quad (n \geq 2),$$ Distribution of m: random

※ Only low-mid wavenumber space is shown for simplicity (energy is almost zero at higher wavenumbers)

Rossby waves 2D incompressible flow on a rotating sphere

Wave solutions called **Rossby waves** (specific for rotating systems)

$\boldsymbol{\Omega} = (0, 0, \Omega)$



$$\begin{cases} Y_n^m(\lambda, \mu)\exp(-i\omega t), \\ \omega = \dfrac{-2m\Omega}{n(n+1)} \end{cases}$$

$$Y_n^m(\lambda, \mu) = P_n^m(\lambda, \mu)\exp(-im\lambda)$$

: spherical harmonics

Dynamics of Rossby waves determines the temporal variation of flow filed.
(Three-wave)Nonlinear interactions of Rossby waves are important!

We investigate **energy accumulation to zonal Rossby modes with low n**
from the perspective of three-Rossby-wave nonlinear interaction

Briefly introduce

- zonal Rossby modes
- conditions for three-wave nonlinear interaction
- three-wave resonant nonlinear interaction

three-wave resonant interaction of Rossby waves

Nonlinear interaction of three Rossby waves $Y_{n_B}^{m_B} \times Y_{n_C}^{m_C} \rightarrow Y_{n_A}^{m_A}$ is **resonant interaction** when they satisfy

Necessary conditions for three-wave interaction

$$m_B + m_C = m_A$$

$$|n_B - n_C| \le n_A \le n_B + n_C$$

$$n_A + n_B + n_C = \text{odd integer}$$

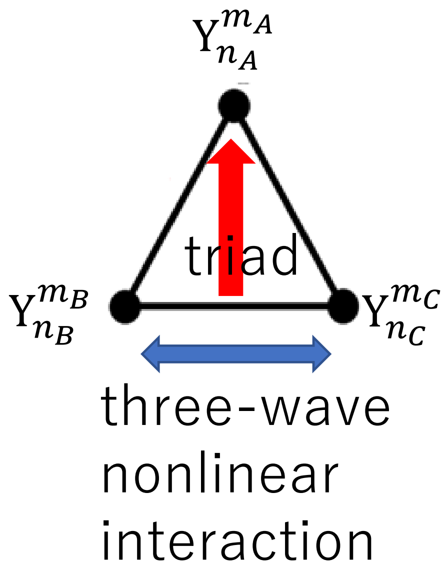
$$n_A, n_B, n_C > 0$$

Additional condition for resonant interaction

$$\frac{m_B}{n_B(n_B+1)} + \frac{m_C}{n_C(n_C+1)} = \frac{m_A}{n_A(n_A+1)}$$

Rossby waves

$$Y_n^m(\lambda, \mu)\exp(i\omega t),\quad \omega = \frac{-2m\Omega}{n(n+1)}$$



three-wave resonant interaction of Rossby waves

- Specifically, in case of $\Omega \to \infty$ (or $\beta \to \infty$)flow dynamics is totally governed by 3-(Rossby) wave resonant nonlinear interactions:

→When Ω is infinite, resonant interactions determines flow dynamics for finite period of time (local existence time)T(Ω).)

Theorem 2. *Assume that* $a(0) := \{a_n(0)\}_{n\in\mathbb{Z}^2} \in \ell_1(\mathbb{Z}^2)$. *Then there is a local existence time* T_L *and a local-in-time unique solution* $a(t) := \{a_n(t)\}_{n\in\mathbb{Z}^2} \in C([0,T_L] : \ell_1(\mathbb{Z}^2))$ *satisfying*

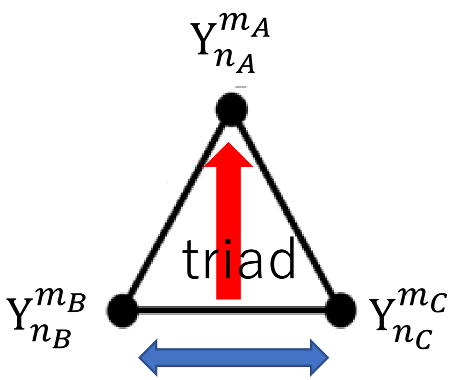
$$T_L \geq \frac{C}{\|a_0\|^2}, \qquad \sup_{0<t<T_L} \|a(t)\| \leq 2\|a_0\|,$$

where C *is a positive constant independent of* β. *Moreover if* $\|a(0)\|_s < \infty$ *for* $s \geq 0$, *then we have the following pointwise estimate:*

$r_n(t)$: remainder term

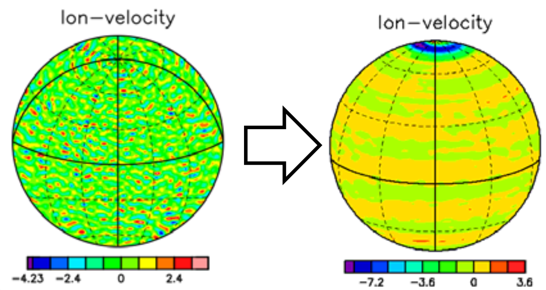
Theorem 3. *For all* $\epsilon > 0$, *there is* $\beta_0 > 0$ *s.t.* $\|r(t)\| \leq \epsilon$ *for* $0 < t < T_L$ *and* $|\beta| > \beta_0$, *where* T_L *is the local existence time (see Theorem 2).*

T. Yoneda and M.Yamada (2013)
A. Dutrifoy and M. Yamada (in preparation)



three-wave nonlinear interaction

Resonant interactions should be the type of interactions that works most strongly in the numerical calculations.



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- Summary and Discussion

What kind of

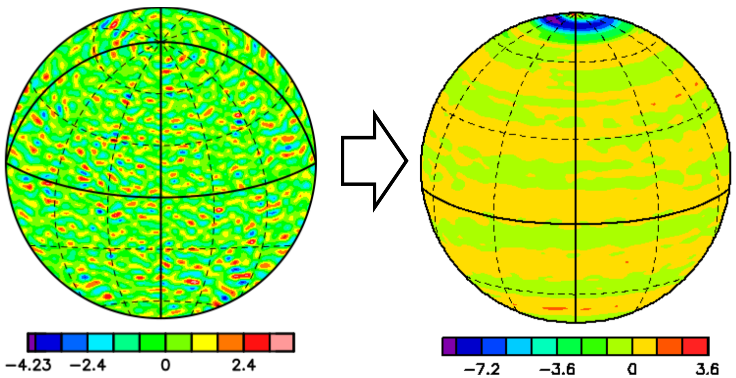
$$Y_{n_B}^{m_B} \times Y_{n_C}^{m_C} \rightarrow Y_{n_A}^{\mathbf{0}}$$

are important?

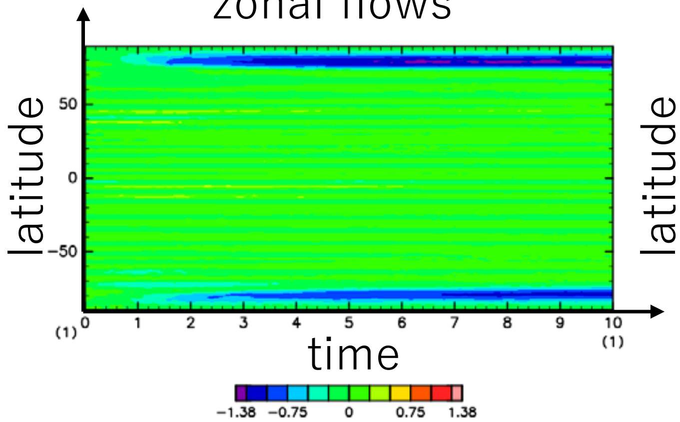
Three-wave nonlinear interactions and zonal flow formation

- westward circumpolar zonal flows (Yoden and Yamada1993, Takehiro et al. 2007)

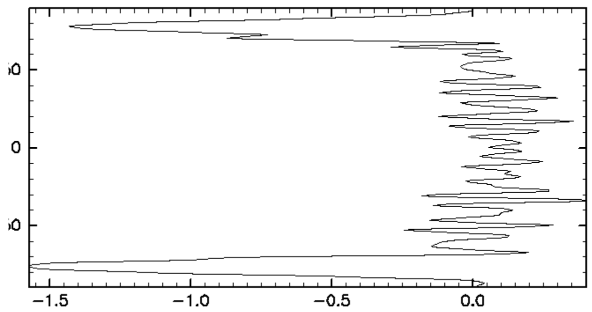
longitudinal-velocity



temporal development of zonal flows



westward circumpolar jets



What kind of three-Rossby-wave nonlinear interactions are the direct factors in the formation of large-scale zonal flows?
→ introduce two types of nonlinear interactions $Y_{n_B}^{m_B} \times Y_{n_C}^{m_C} \rightarrow Y_{n_A}^{0}$:
Near-Resonant interactions and **Non-Local** interactions

Near-Resonant nonlinear interactions and large-scale zonal flow formation

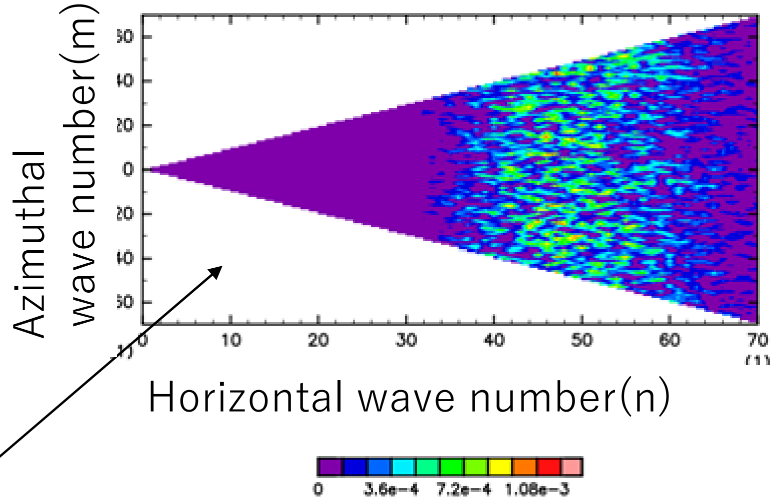
Development of zonal Rossby modes

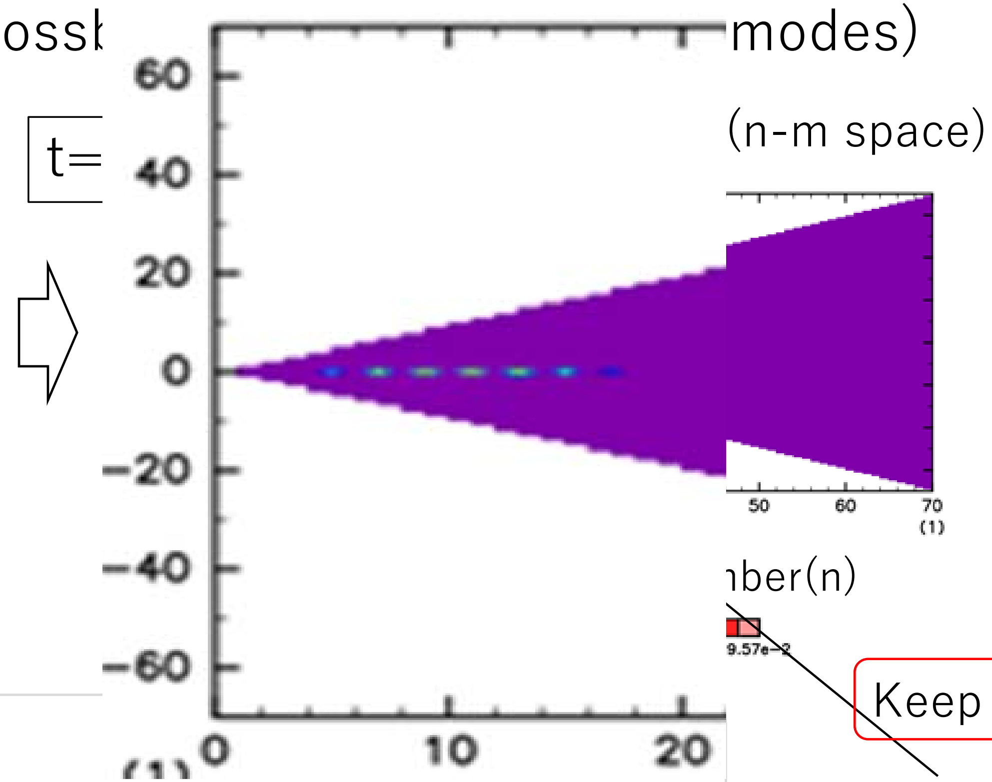
Development of zonal flow ↙ Spherical harmonics with zero upper subscripts have a zonal structure

= energy accumulation to $Y_n^0(\lambda, \mu)$ Rossb… modes)

t=0

energy spectrum E_n^m(n-m space)





Random initial condition

$$E(n, t=0) = \frac{An^{\gamma/2}}{(n+n_0)^{\gamma}},\quad n_0 = 50,\quad \gamma = 100,\quad (n \geq 2),$$ Distribution of m: random

Keep in mind

Correspond to circumpolar zonal flows

※ Only low-mid wavenumber space is shown for simplicity (energy is almost zero at higher wavenumbers)

Energy of zonal Rossby modes E_n^0

Sum of energy of zonal modes $\sum_{n:odd} E_n^0(\lambda,\mu)$, $\sum_{n:even} E_n^0(\lambda,\mu)$

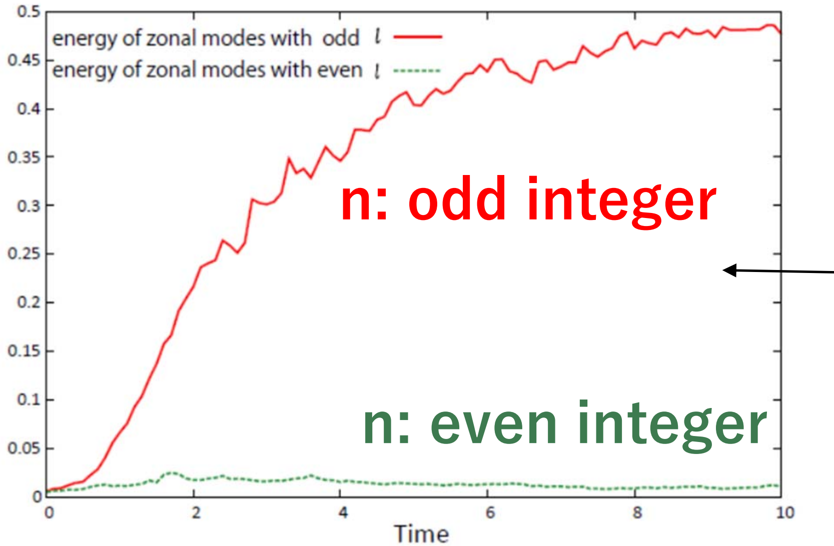
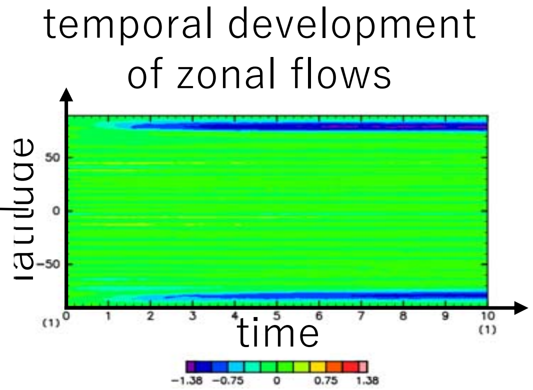
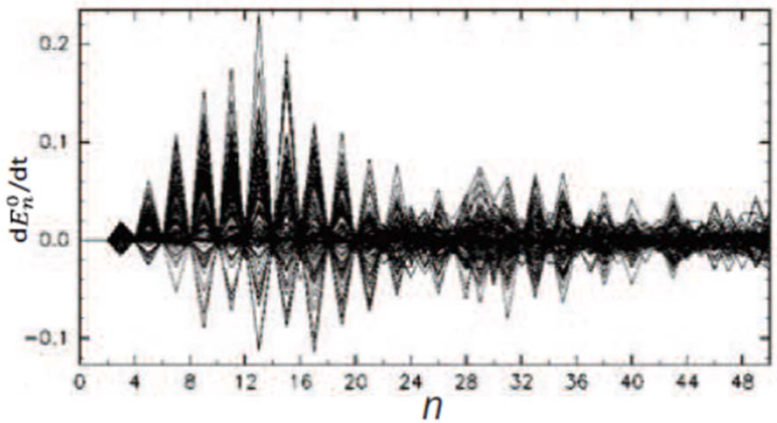


Fig. 3. (Color online) Temporal variation of the $Y^0_{l=odd}$ and $Y^0_{l=even}$ modes from $t = 0$ to 10 in $\nu_{2p} = 0.0$ case.





Energy is only accumulated to zonal modes whose n is low and odd integer $Y^0_{n:\mathrm{odd}}$

(O. and Yamada, 2020)

Keep in mind

→ a clue to the elucidation of the zonal flow formation mechanism?

Speaking of the dependence on the parity of n, ⇩

one important point we can think of is the three-wave resonant interaction.

three-wave resonant interaction of Rossby waves

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Rossby waves

$$Y_n^m(\lambda, \mu)\exp(i\omega t),\quad \omega = \frac{-2m\Omega}{n(n+1)}$$

Necessary conditions for three-wave interaction

$$m_B + m_C = m_A$$

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Additional condition for resonant interaction

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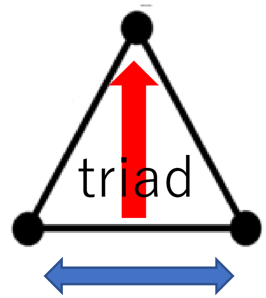


three-wave nonlinear interaction

$Y_{n=odd}^0$ can be a member of resonant triads (**resonant zonal modes**)
$Y_{n=even}^0$ are not resonant zonal modes

Energy accumulation to resonant zonal modes

Zonal Rossby waves and tree-wave resonant interaction

- Specifically, in case of $\Omega \to \infty$ (or $\beta \to \infty$)flow dynamics is totally governed by 3-(Rossby) wave resonant nonlinear interactions:

→When Ω is infinite, resonant interactions determines flow dynamics for finite period of time (local existence time)T(Ω).)

Theorem 2. *Assume that* $a(0) := \{a_n(0)\}_{n\in\mathbb{Z}^2} \in \ell_1(\mathbb{Z}^2)$. *Then there is a local existence time* T_L *and a local-in-time unique solution* $a(t) := \{a_n(t)\}_{n\in\mathbb{Z}^2} \in C([0,T_L] : \ell_1(\mathbb{Z}^2))$ *satisfying*

$$T_L \geq \frac{C}{\|a_0\|^2}, \qquad \sup_{0<t<T_L} \|a(t)\| \leq 2\|a_0\|,$$

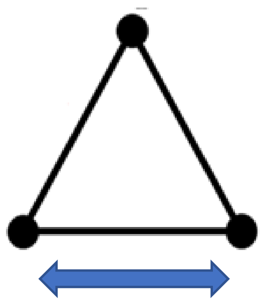
where C *is a positive constant independent of* β. *Moreover if* $\|a(0)\|_s < \infty$ *for* $s \geq 0$, *then we have the following pointwise estimate:*

$r_n(t)$: remainder term

Theorem 3. *For all* $\epsilon > 0$, *there is* $\beta_0 > 0$ *s.t.* $\|r(t)\| \leq \epsilon$ *for* $0 < t < T_L$ *and* $|\beta| > \beta_0$, *where* T_L *is the local existence time (see Theorem 2).*

T. Yoneda and M.Yamada (2013)
A. Dutrifoy and M. Yamada (in preparation)

triad



three-wave nonlinear interaction

- **resonant interaction is dominant**
- **only resonant modes ($Y^0_{n=odd}$) develop**

⇩

Can the formation of zonal flows be explained only by three-wave nonlinear resonant interactions? **NO**

three-wave resonant interaction of Rossby waves

Nonlinear interaction of three Rossby waves $Y_k^q \times Y_m^r \rightarrow Y_n^p$ is **resonant interaction** when they satisfy

Conditions for three-wave interaction

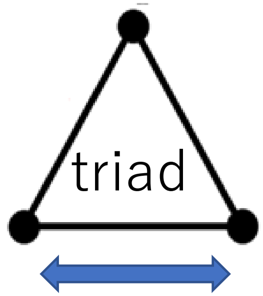
$$\begin{cases} m_B + m_C = m_A \\ |n_B - n_C| \leq n_A \leq n_B + n_C \\ n_A + n_B + n_C = \text{odd integer} \\ n_A, n_B, n_C > 0 \end{cases}$$

Additional condition for resonant interaction →

$$\frac{m_B}{n_B(n_B+1)} + \frac{m_C}{n_C(n_C+1)} = \frac{m_A}{n_A(n_A+1)}$$

Rossby waves

$$\begin{cases} Y_n^m(\lambda, \mu) \exp(i\omega t), \\ \omega = \dfrac{-2m\Omega}{n(n+1)} \end{cases}$$



three-wave nonlinear interaction

$Y^0_{n=odd}$ can be a member of resonant triads (**resonant zonal modes**)
$Y^0_{n=even}$ are not resonant zonal modes

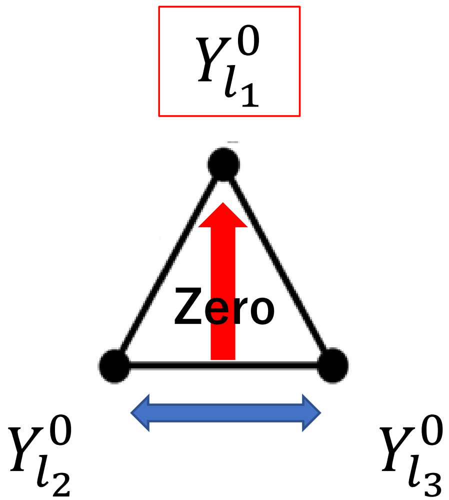
No energy is transferred to zonal modes by resonant interactions
(Reznik et.al. 1993, Obuse and Yamada 2019)

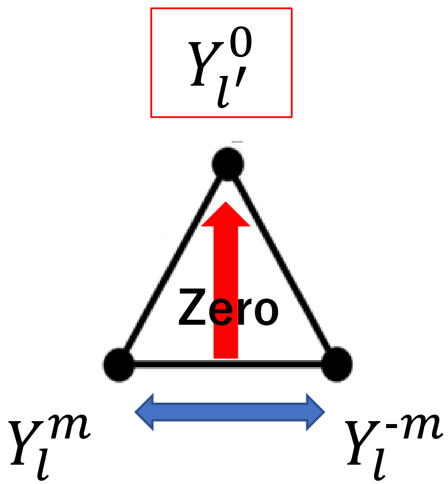
No energy transfer to zonal Rossby modes by resonant interaction

Resonant triad with three zonal Rossby modes:

(Obuse and Yamada 2019)

$$Y_{l_2}^{0} \times Y_{l_3}^{0} \rightarrow Y_{l_1}^{0} \qquad Y_{l}^{m} \times Y_{l}^{-m} \rightarrow Y_{l'}^{0}$$



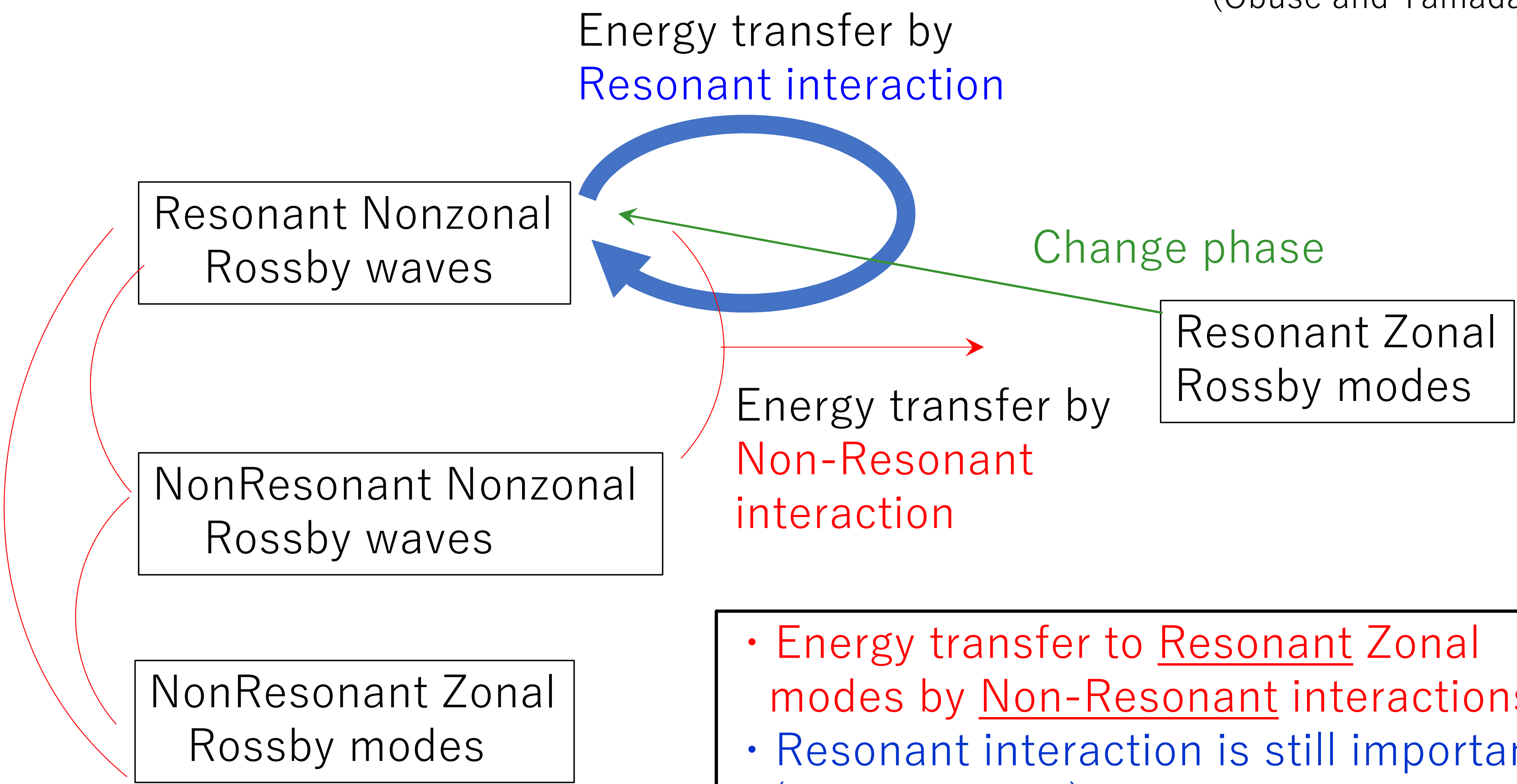


$$\boxed{Y_{l_2}^{m_2} \times Y_{l_3}^{m_3} \rightarrow Y_{l_1}^{m_1}}$$

$$\begin{aligned} &J\left(\psi_{l_2}^{m_2} Y_{l_2}^{m_2}, l_3(l_3+1)\psi_{l_3}^{m_3} Y_{l_3}^{m_3}\right) + J\left(\psi_{l_3}^{m_3} Y_{l_3}^{m_3}, l_2(l_2+1)\psi_{l_2}^{m_2} Y_{l_2}^{m_2}\right) \\ &= \psi_{l_2}^{m_2}\psi_{l_3}^{m_3} l_3(l_3+1)\left(\frac{\partial Y_{l_2}^{m_2}}{\partial \phi}\frac{\partial Y_{l_3}^{m_3}}{\partial \mu} - \frac{\partial Y_{l_2}^{m_2}}{\partial \mu}\frac{\partial Y_{l_3}^{m_3}}{\partial \phi}\right) \\ &+ \psi_{l_2}^{m_2}\psi_{l_3}^{m_3} l_2(l_2+1)\left(\frac{\partial Y_{l_3}^{m_3}}{\partial \phi}\frac{\partial Y_{l_2}^{m_2}}{\partial \mu} - \frac{\partial Y_{l_3}^{m_3}}{\partial \mu}\frac{\partial Y_{l_2}^{m_2}}{\partial \phi}\right), \end{aligned}$$

No energy transfer to zonal Rossby modes by resonant interaction

(Obuse and Yamada 2019, 2020)



- Energy transfer to Resonant Zonal modes by Non-Resonant interactions
- Resonant interaction is still important (and dominant).

Zonal Rossby waves and three-wave resonant interaction

Nonlin
resona

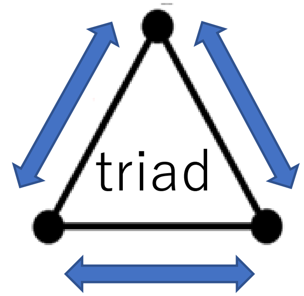
ssby waves
$(\lambda, \mu) \exp(i\omega t)$,
$= \dfrac{-2m\Omega}{n(n+1)}$

Condit
three-
interac

Then, how about near-resonant interactions??

Additional condition for resonant interaction

$$\frac{m_B}{n_B(n_B+1)} + \frac{m_C}{n_C(n_C+1)} \sim \frac{m_A}{n_A(n_A+1)}$$



three-wave nonlinear interaction

$Y^0_{n=odd}$ can be a member of resonant triads (**resonant zonal modes**)
$Y^0_{n=even}$ are not resonant zonal modes

No energy is transferred to zonal modes by resonant interactions
(by detailed balance energy) (Reznik et.al. 1993, O. and Yamada 2019)

Do near-resonant interactions transfer energy to circumpolar zonal flow?

Consider three-wave interaction by nonlinear term

wave A ($Y_{n_B}^{m_B}$) × wave B ($Y_{n_C}^{m_C}$) → wave C ($Y_{n_A}^{0}$)

resonant interaction : $\omega_A - (\omega_B + \omega_C) = 0$

Define near-resonant interactions as

When $\omega_A - (\omega_B + \omega_C) \leq \varepsilon \times Ro$,

where Ro: Rossby number

$Ro = \frac{U}{2\Omega L}$, $L^{-1} = n_{(\text{energy wighted mean})} = \frac{\sum_n nE_n}{\sum_n E_n}$, $U = \sqrt{2E}$

$\varepsilon = 1.1 \times 2\Omega$

(Similar definition to Smith and Lee 2005)

Then see $\left[\frac{dE_{n_A}^{0}}{dt}\right]_{Near-R} = \sum_{\text{All possible near-resonant triads}}$ Effect of three-wave interaction

$\left[\frac{dE_{n_A}^{0}}{dt}\right]_{Non-R} = \sum_{\text{All possible non-resonant triads}}$ Effect of three-wave interaction

Time variation of dE_n^0/dt

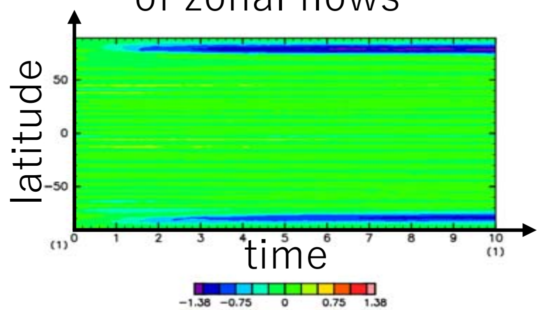
Time variation of energy of zonal mode $\frac{dE_n^0}{dt}$
by nonlinear interactions of Rossby waves

$$\frac{dE_n^0}{dt} = n(n+1)\boxed{\frac{d\psi_n^0}{dt}}\psi_n^0$$

$$\psi(t,\lambda,\mu) = \sum_{n=0}^{N}\sum_{m=-n}^{n} \underline{\psi_n^m(t)}\, Y_n^m(\lambda,\mu)$$

Use flow field data

temporal development
of zonal flows



Coefficients of nonlinear interactions

Time derivative of stream function spectrum:

$$\psi(t,\lambda,\mu) = \sum_{n=0}^{N}\sum_{m=-n}^{n} \underline{\psi_n^m(t)}\, Y_n^m(\lambda,\mu)$$

$$\frac{\partial \psi_n^m(t)}{\partial t} = \frac{2im\psi_n^m(t)}{n(n+1)} + \frac{i}{2}\sum_{s=0}^{N}\sum_{r=-s}^{s}\sum_{k=0}^{N}\sum_{j=-k}^{k} \psi_k^j(t)\psi_s^r(t) H_{kns}^{jmr}$$

from linear terms

from nonlinear terms

nonlinear interaction coefficients of $Y_k^j \times Y_s^r \rightarrow Y_n^m$

$$H_{kns}^{j0-j} = \frac{s(s+1)-k(k+1)}{n(n+1)} L_{kns}^{j0-j}$$

$$L_{kns}^{j0-j} = (-1)^j L_{nks}^{0jj}$$

$$= (-1)^j \{E_{nks}^{0jj} - E_{skn}^{jj0}\}$$

$$= (-1)^j E_{nks}^{0jj}$$

$$E_{nks}^{0jj} = j\sqrt{2n+1}\sum_q \sqrt{2q+1}\int_0^\pi P_q^0 P_k^j P_s^j \sin\theta d\theta$$

necessary conditions for nonzero H_{kns}^{j0-j}:

- $q = n-1, n-3, n-5, \cdots, 1 \text{ or } 0$
- $n+k+s =$ odd integer
- $|n-s| < k < n+s$ (Silberman, 1954)

$$\int_0^\pi P_a^b P_c^d P_e^f \sin\theta d\theta$$

$$= \frac{(e+a-c-1)!![(2c+1)(2a+1)(2e+1)]^{\frac{1}{2}}}{(e+c-a)!!(a+c-e)!!(c+a+e+1)!!}$$

$$\times \left[\frac{(c+d)!(c-d)!(a-b)!(e-f)!}{2(a+b)!(e+f)!}\right]^{\frac{1}{2}}$$

$$\times \sum_{h=0}^{c-d} \frac{(-1)^{\frac{1}{2}(e-a+c)+f+h}(e+f+h)!(a+c-f-h)!}{(c-d-h)!h!(e-f-h)!(a-c+f+h)!}$$

(Hull 1951)

Time variation of dE_n^0/dt

Time variation of energy of zonal mode $\frac{dE_n^0}{dt}$
by nonlinear interactions of Rossby waves

$$\frac{dE_n^0}{dt} = n(n+1)\boxed{\frac{d\psi_n^0}{dt}}\,\psi_n^0$$

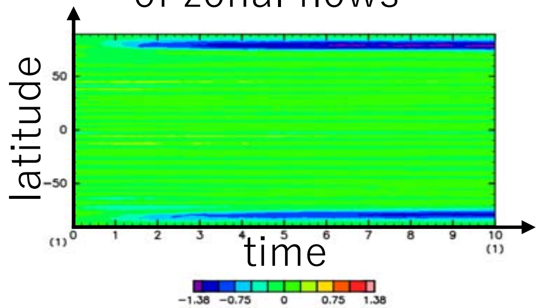
$$\psi(t,\lambda,\mu) = \sum_{n=0}^{N}\sum_{m=-n}^{n} \underline{\psi_n^m(t)}\, Y_n^m(\lambda,\mu)$$

Use flow field data

$$\boxed{\frac{\partial \psi_n^m(t)}{\partial t} = \frac{2im\psi_n^m(t)}{n(n+1)} + \frac{i}{2}\sum_{s=0}^{N}\sum_{r=-s}^{s}\sum_{k=0}^{N}\sum_{j=-k}^{k} \psi_k^j(t)\psi_s^r(t)H_{kns}^{jmr}}$$

Use nonlinear interaction coefficients
calculated from Silberman' notation

temporal development
of zonal flows



Do near-resonant interactions transfer energy to circumpolar zonal flow?

Consider three-wave interaction by nonlinear term

wave A ($Y_{n_B}^{m_B}$) × wave B ($Y_{n_C}^{m_C}$) → wave C ($Y_{n_A}^{\mathbf{0}}$)

resonant interaction :
$\omega_A - (\omega_B + \omega_C) = 0$

Define near-resonant interactions as

When $\omega_A - (\omega_B + \omega_C) \leq \varepsilon \times \mathrm{R}o$,

where
$\mathrm{R}o$: Rossby number
$\mathrm{R}o = \frac{\mathrm{U}}{2\Omega \mathrm{L}}$, $\mathrm{L}^{-1} = n_{(\text{energy wighted mean})} = \frac{\sum_n nE_n}{\sum_n E_n}$, $\mathrm{U} = \sqrt{2E}$
$\varepsilon = 1.1 \times 2\Omega$

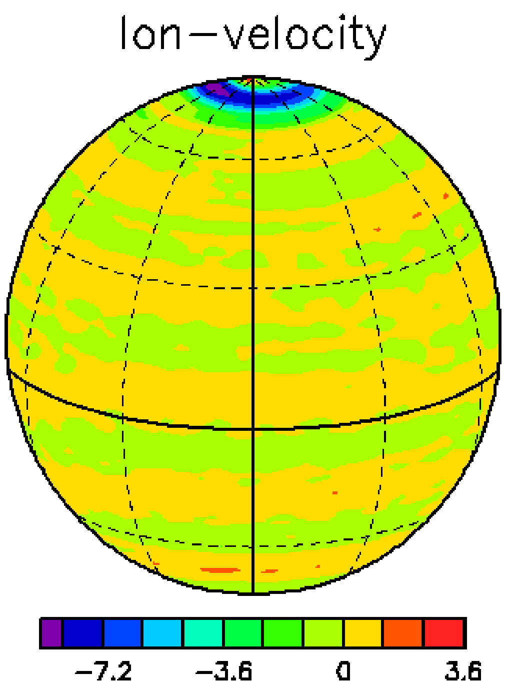
(Similar definition to Smith and Lee 2005)

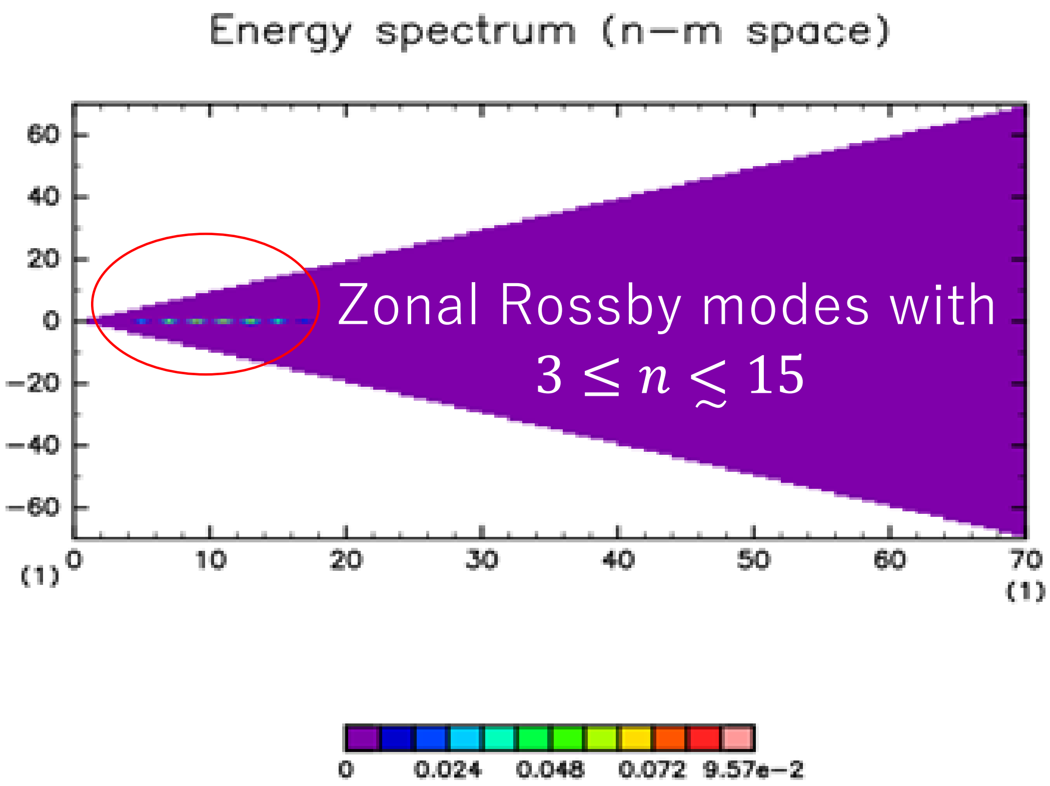
Then see $\left[\frac{dE_{n_A}^0}{dt}\right]_{Near-R} = \sum_{\text{All possible near-resonant triads}}$ Effect of three-wave interaction

$\left[\frac{dE_{n_A}^0}{dt}\right]_{Non-R} = \sum_{\text{All possible non-resonant triads}}$ Effect of three-wave interaction

Do near-resonant interactions transfer energy to circumpolar zonal flow?

Before seeing the result, please recall





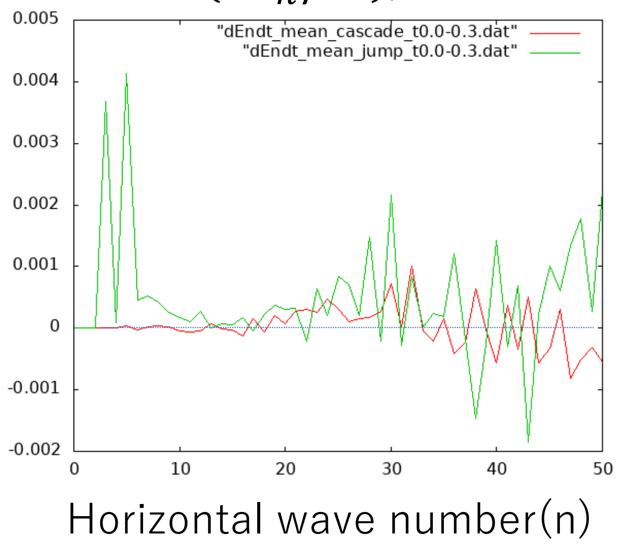
Do near-resonant interactions transfer energy to circumpolar zonal flow?

mean $\left(\frac{dE_n^0}{dt}\right)$ caused by near-resonant and non-resonant interactions

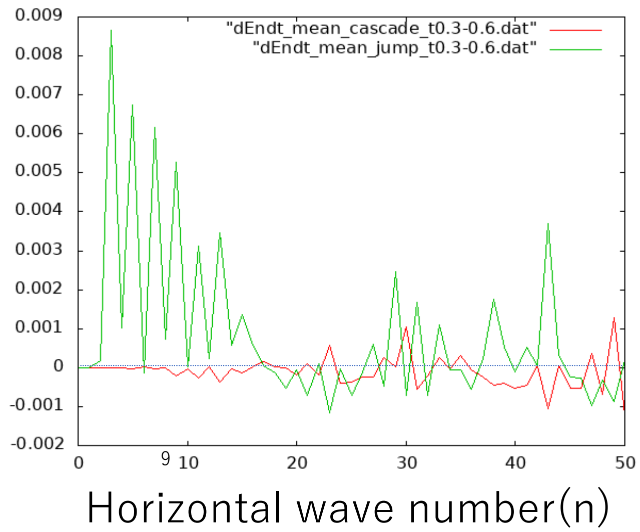
$$(\omega_A - (\omega_B + \omega_C) \lesssim 1.1Ro)$$

※Red: Non-Resonant, Green: Near-Resonant

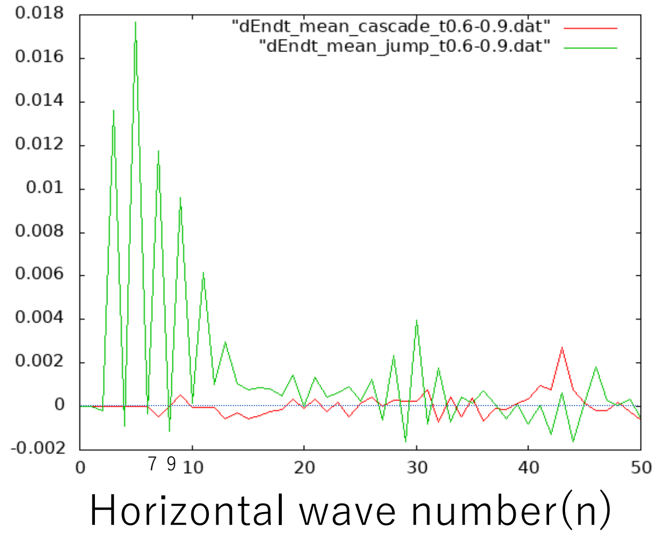
mean(dE_n^0/dt),t=0.0-0.3



mean(dE_n^0/dt),t=0.3-0.6



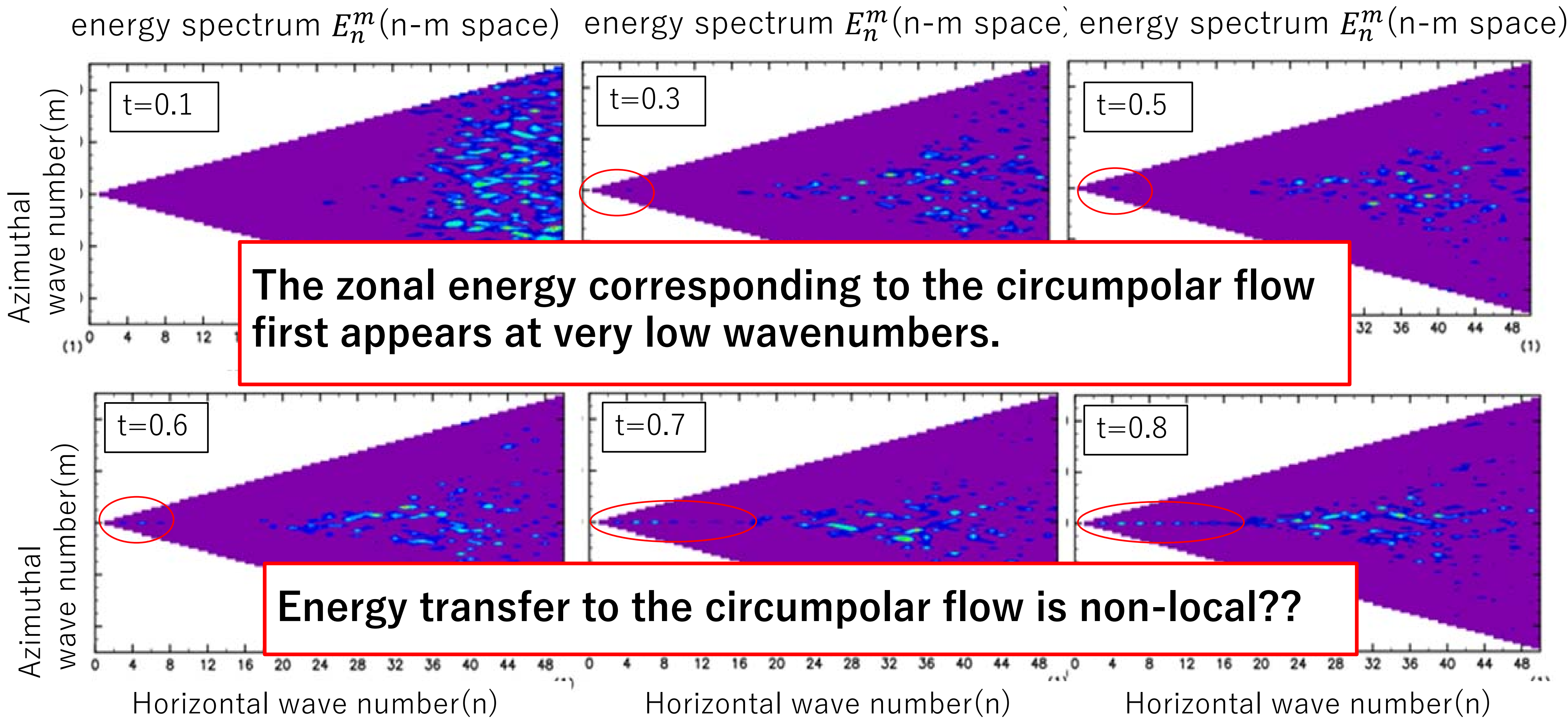
mean(dE_n^0/dt),t=0.6-0.9



Only Near-Resonant interactions are directly sending energy to circumpolar zonal flows.

Non-Local nonlinear interactions and large-scale zonal flow formation

Energy spectrum (energy of Rossby waves) E_n^m at certain times

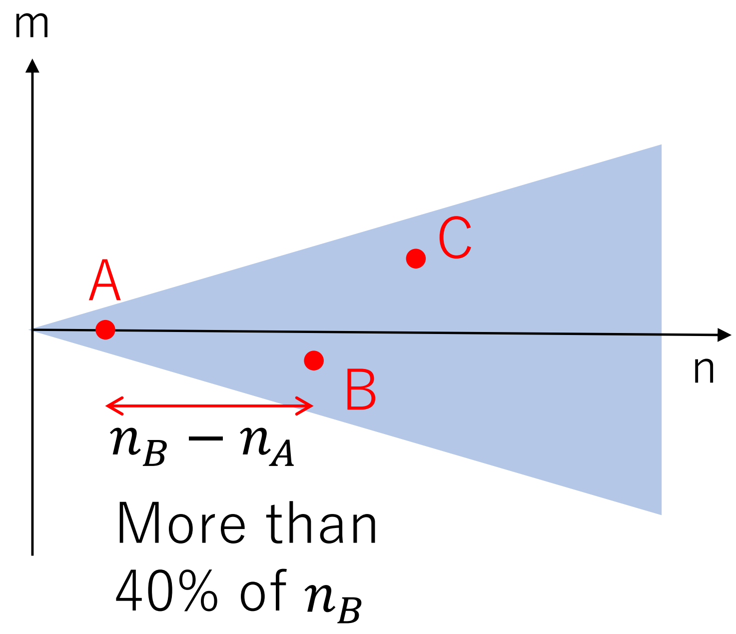


※ wavenumber region only n=0-50 is shown for simplicity

Is energy transfer to circumpolar flow non-locally?

Consider three-wave interaction by nonlinear term

wave A ($Y_{n_B}^{m_B}$) × wave B ($Y_{n_C}^{m_C}$) →wave C ($Y_{n_A}^{\mathbf{0}}$)



Define the non-local interactioin as

When $\min(n_B, n_C) - n_A \geq \mathrm{N\%}$ of $\min(n_B, n_C)$,
N=40

Then see $\left[\frac{dE_{n_A}^0}{dt}\right]_{Non-Local} = \sum_{\text{All possible non-local interactions}}$ Effect of three-wave interaction

$\left[\frac{dE_{n_A}^0}{dt}\right]_{Local} = \sum_{\text{All possible local interactions}}$ Effect of three-wave interaction

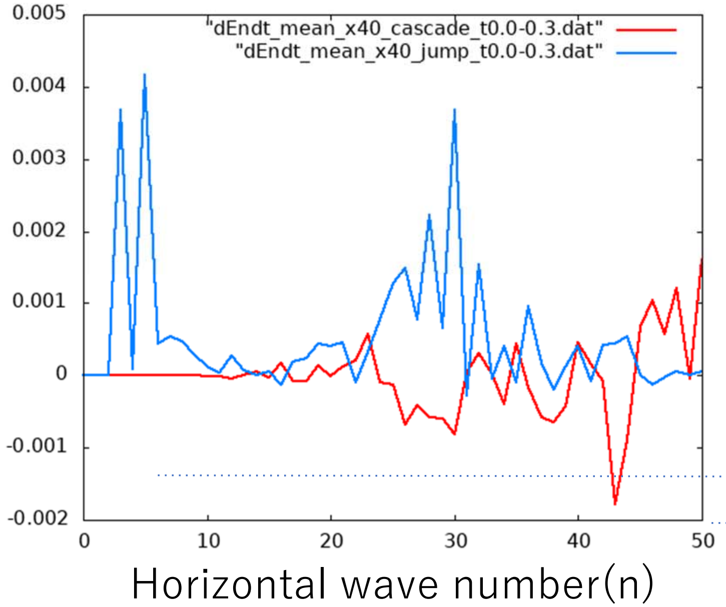
Is energy transfer to circumpolar flow mainly nonlocal?

mean $\left(\frac{dE_n^0}{dt}\right)$ caused by local and non-local inetrcations

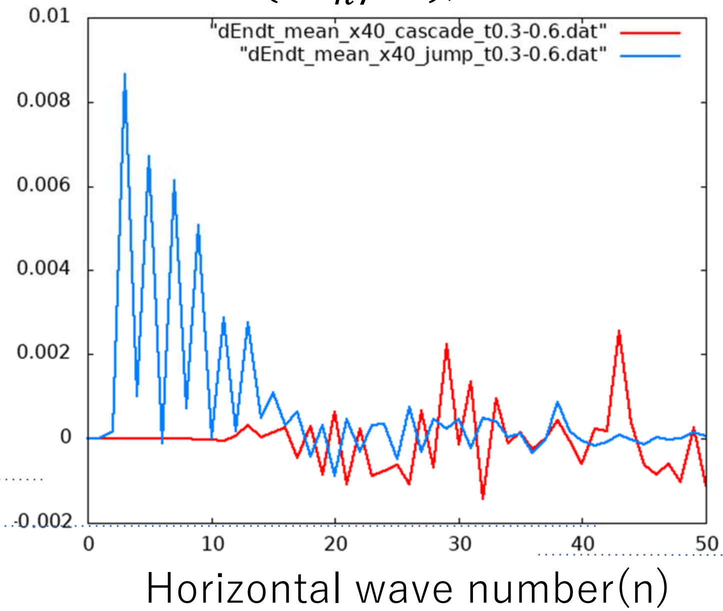
※Red: Local, Blue: Non-Local

$$(\min(n_B, n_C) - n_A \gtrless \min(n_B, n_C) \times \frac{40}{100})$$

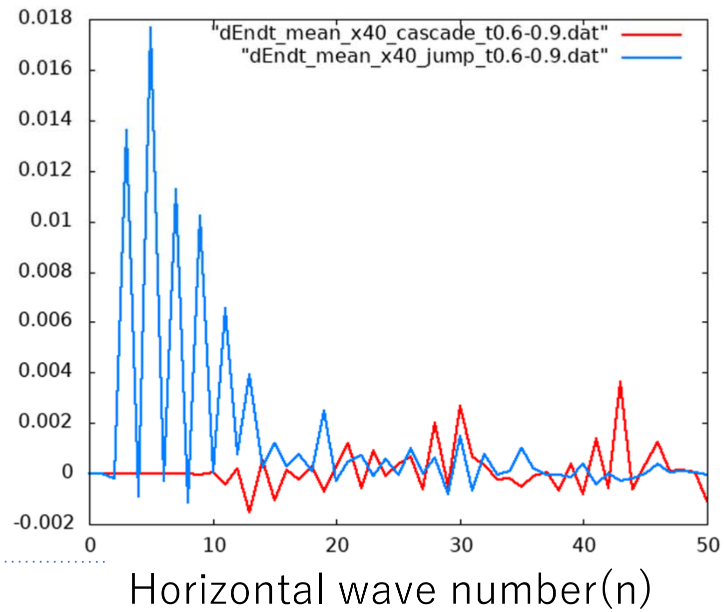
mean(dE_n^0/dt),t=0.0-0.3



mean(dE_n^0/dt),t=0.3-0.6



mean(dE_n^0/dt),t=0.6-0.9



Only Non-Local interactions are directly sending energy to circumpolar zonal flows.

Non-Local Near-Resonant nonlinear interactions and large-scale zonal flow formation

Non-Local Near-Resonant nonlinear interactions and large-scale zonal flow formation

Near-Resonant interaction and Non-Local interaction

are completely different concepts

We see mean $\left(\frac{dE_n^0}{dt}\right)$ caused by

- ✓ Non-Local Near-Resonant,
- ✓ Non-Local Non-Resonant
- ✓ Local Near-Resonant,
- ✓ Local Non-Resonant

nonlinear interactions

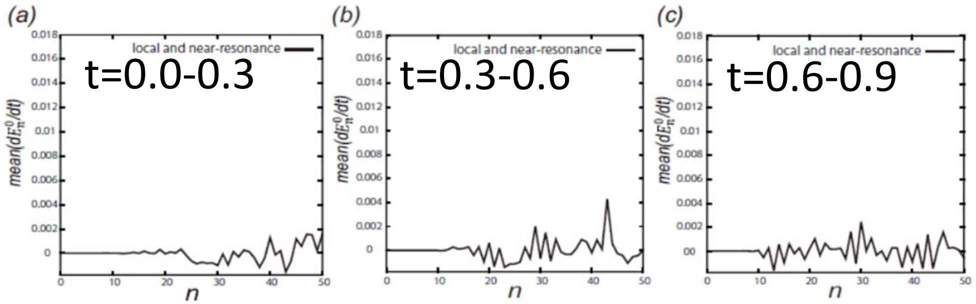
Non-Local Near-Resonant nonlinear interactions and large-scale zonal flow formation

Near-Resonant interaction and Non-Local interaction
are completely different concepts

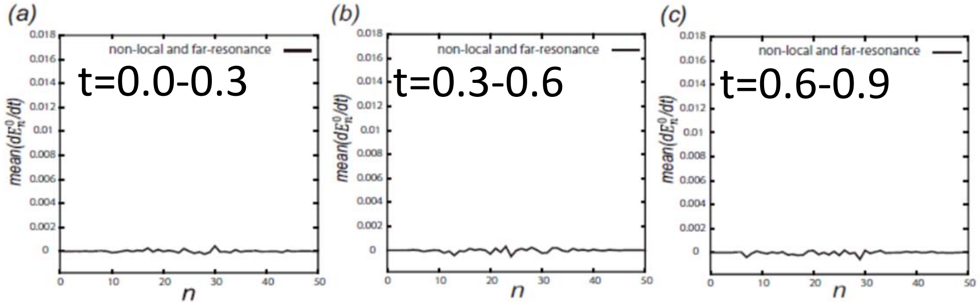
---

mean $\left(\frac{dE_n^0}{dt}\right)$ caused by Non-Local Non-Resonant, Local Near-Resonant,
and Local Near-Resonant interactions

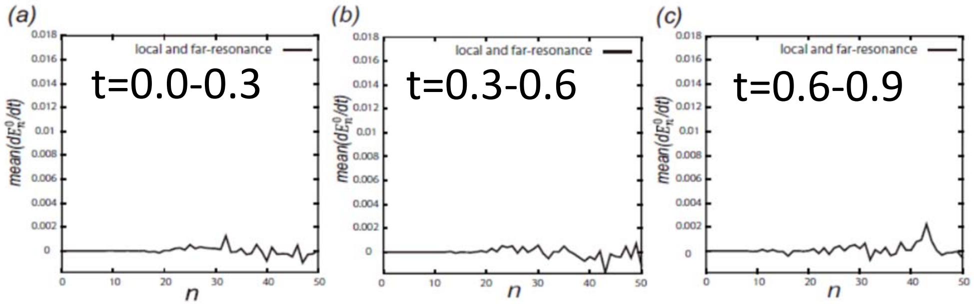
Local・Near-Resonant



Non-Local・Non-Resonant



Local・Non-Resonant



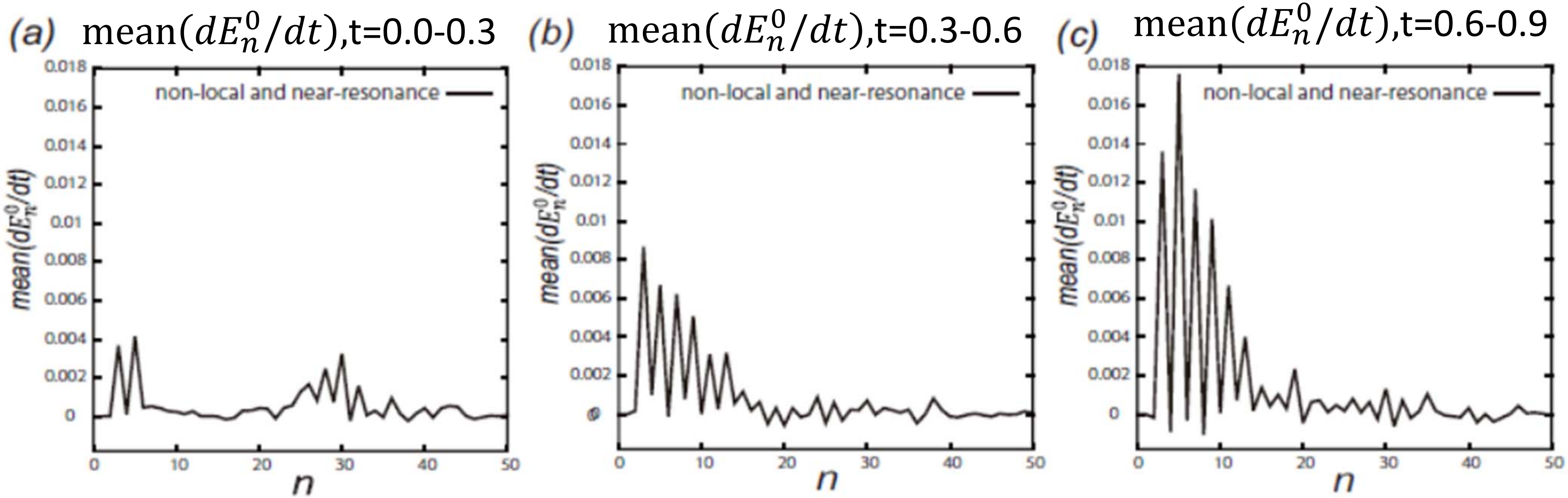
Energy transfer
to large-scale zonal modes
by these interactions is almost zero.

Non-Local Near-Resonant nonlinear interactions and large-scale zonal flow formation

Near-Resonant interaction and Non-Local interaction are completely different concepts

---

mean $\left(\frac{dE_n^0}{dt}\right)$ caused by Non-Local Near-Resonant interactions



{Near-Resonant interactions}~{Non-Local interactions} in this system, and energy is non-locally transferred to large-scale zonal modes by near-resonant interactions

Summary and Discussion

In unforced two-dimensional turbulence on a rotating sphere
→ formation of large-scale circumpolar westward zonal flows,
but the mechanism of zonal flow formation is not well understood.

By closely seeing energy transfer, it was suggested that
direct factor of large-scale zonal flow formation is

Non-Local energy transfer to large-scale zonal modes
by Near-Resonant interactions

Summary and Discussion

Direct factor of large-scale zonal flow formation is

Non-Local energy transfer to large-scale zonal modes by Near-Resonant interactions

Importance of **non-local interactions** may be strongly related to the **compactness** of the considered flow field domain. Spherical domain is compact and **basic modes are discrete**. Therefore the nonlinear interactions must be non-local, and this tendency is stronger in the low wavenumber region.

In numerical calculations, not only does it happen that the nonlocal near-resonant interaction works strongly due to the geometric constraint, but the **flow field evolves in time from a random uniform initial state so that the non-local near-resonant interaction dominates.**

Importance of **near-resonant interactions** is in **natural agreement with the intuition** derived from the fact that resonant interactions are the ones that work most strongly in this system, but do not function on zonal Rossby modes.