Characterising Edge States:

Measures on chaotic non-attracting invariant sets

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For attractors, there is the concept of physical measures which quantify the probability of a trajectory to be at a certain location on the attractor during a given time interval. A similar measure on saddles and repellers (both are non-attracting invariant sets) would be desirable as it contains information about their geometry, and about possible long-living chaotic transients, which helps quantifying the level of uncertainty of the long-term behaviour of the system. Chaotic transients can occur in reality, e.g. in turbulent flows, or even in simple models of the earth's climate.

In this work, we explore an invariant probability measure on non-attracting invariant sets that under certain conditions is a candidate for a physical measure on these sets. We explore its properties and analogous measures on the invariant set's stable and unstable set.

Importance of measures on chaotic saddles in climate science

- A physical measure on an attractor tells us which states are likely to be realised and which are not. For example in climate science, such a measure might tell us how probable a specific weather condition is.
- A system can be in a state on or close to a chaotic saddle for a long time, and a physical measure on this saddle would characterise the dynamics of the transient dynamics on or close to the saddle. Such a measure also contains information on the geometric complexity and fractal dimension of the saddle. In the special case of the saddle being an edge state (an attractor of the system restricted to the basin boundary) this also contains information about the dimension of the basin boundary and thus, about uncertainty close to it.

Short summary:

- We introduce a novel notion of an invariant probability measure on chaotic non-attracting invariant sets in terms of Lebesgue measure.
- We explain how to approximate the measures by "uniform sprinkling" according to the background measure, and we use the central limit theorem to find the convergence rate of the approximation.
- We show a numerical example where we approximate the measure on a saddle and on its stable and unstable set.
- We use a formula from [1] to compute the Information dimension of the saddle from its finite-time Lyapunov exponents. We find it to be very close to the box-counting dimension of the support of the previously computed measure on the saddle.
- We show that the measures on the stable and unstable set defined following [1] make sense if the system is Lebesgue invariant.

Definitions

Inspired by Sweet and Ott [1], we start by considering a region R that contains the non-attracting invariant set Λ, but no other invariant set.

We define the region of trajectories that remained within R up until n iterations as

$$R^{(n)} = \{x \in R : f^k(x) \in R, k = 0, ..., n\} \tag{1}$$

$$= \bigcap_{k=0}^{n} f^{-k}(R). \tag{2}$$

Note that $R^{(n+1)} \subset R^{(n)}$ is a nested sequence.

We further define the set of trajectories that have been in the open set $C \subset R$ after m iterations and which also remained within R up until n iterations as

$$R^{(m,n,C)} = \{x \in R : f^k(x) \in R, k = 0, ..., n \;\&\; f^m(x) \in C\} \tag{3}$$

$$= R^{(n)} \cap f^{-m}(C) \tag{4}$$

with $m \leq n$. Now, we can define the *characteristic escape time* τ from the region R as:

$$\tau^{-1} = -\lim_{n\to\infty} \frac{1}{n} \ln \left(\frac{l(R^{(n)})}{l(R)} \right) \tag{5}$$

And we define the pre-measure μ on Λ for any open set $C \subset R$ by

$$\mu(C) = \lim_{n\to\infty} \frac{l(R^{(m(n),n,C)})}{l(R^{(n)})}, \tag{6}$$

where $m(n)$ is called a mediating sequence and is such that $0 \ll m(n) \ll n$ as $n \to \infty$. We call Λ a compliant maximal non-attracting invariant set if the limit in eq. (6) exists and if it is independent of the mediating sequence $m(n)$. Using Caratheodory's extension theorem, we can extend the pre-measure on the ring of open sets to a measure on all Borel sets. Note that $\mu(C)$ can be proven to be a probability measure supported on Λ.

Approximation through uniform sprinkling

To numerically approximate this measure $\mu(C)$, we show that it is equivalent to the method of Sweet and Ott on approximating Lebesgue measure by uniform sprinkling.

We define the random variable:

$$\mathbf{x} = \{x(i)\}_{i=1}^{\infty} \tag{7}$$

with $x(i) \in R$ chosen uniformly and independently with respect to l.

Define the product measure $L_{R^{(n)}}$

$$L_{R^{(n)}}(\{c(i)\}_{i=1}^{\infty}) = \prod_{i=1}^{\infty} \ell_{R^{(n)}}(c(i)) \tag{8}$$

with $\ell_{R^{(n)}}(C) = \ell(R^{(n)} \cap C)/\ell(R^{(n)})$ and $\{c(i)\}_{i=1}^{\infty}$ is a sequence of open sets. Note that $L_{R^{(n)}}(R^{\mathbb{N}}) = 1$ which means that $L_{R^{(n)}}$ is a probability measure, and

$$\ell_{R^{(n)}}(R^{(m,n,C)}) = \frac{\ell(R^{(m,n,C)})}{\ell(R^{(n)})}. \tag{9}$$

Now define

$$N(\mathbf{x}, m, n, C, k) = \#\{i : 1 \leq i \leq k \text{ and } x(i) \in R^{(m,n,C)}\}, \tag{10}$$

i.e. the number of points in the sequence $x(i), ..., x(k)$ that remain in $R^{(n)}$ up to the nth iterate and that are in C after m iterates.

Theorem 1 *The measure μ in (10) can be computed as*

$$\mu(C) = \lim_{n\to\infty} \lim_{k\to\infty} \frac{N(\mathbf{x}, m(n), n, C, k)}{k} \tag{11}$$

as long as the limits on both sides exist. In this is the case the right-hand side converges for $L_{R^{(n)}}$-almost all choices of $\mathbf{x}$ to the left-hand side.

Proof: The law of large numbers implies that

$$\lim_{k\to\infty} \frac{N(\mathbf{x}, m, n, C, k)}{k} = \ell_{R^{(n)}}(R^{(m,n,C)}). \tag{12}$$

and in particular, the left hand side converges for $L_{R^{(n)}}$-a.a. sequences $\mathbf{x}$ if the limit. Hence we have this equality for a full $L_{R^{(n)}}$-measure set of $\mathbf{x}$. Applying this for each n and intersecting the allowable full measure sets (which still has full measure) gives the required result, as long as the limit in (11) converges. In this sense, the notion of uniform sprinkling in [1] gives an accurate estimate of μ for a.a. choices of $\mathbf{x}$.

For fixed m and n, one can use the Central Limit Theorem for equation (12) and the convergence rate is $\frac{1}{\sqrt{k}}$ to get estimates of how many sample trajectories are needed to estimate μ accurately.

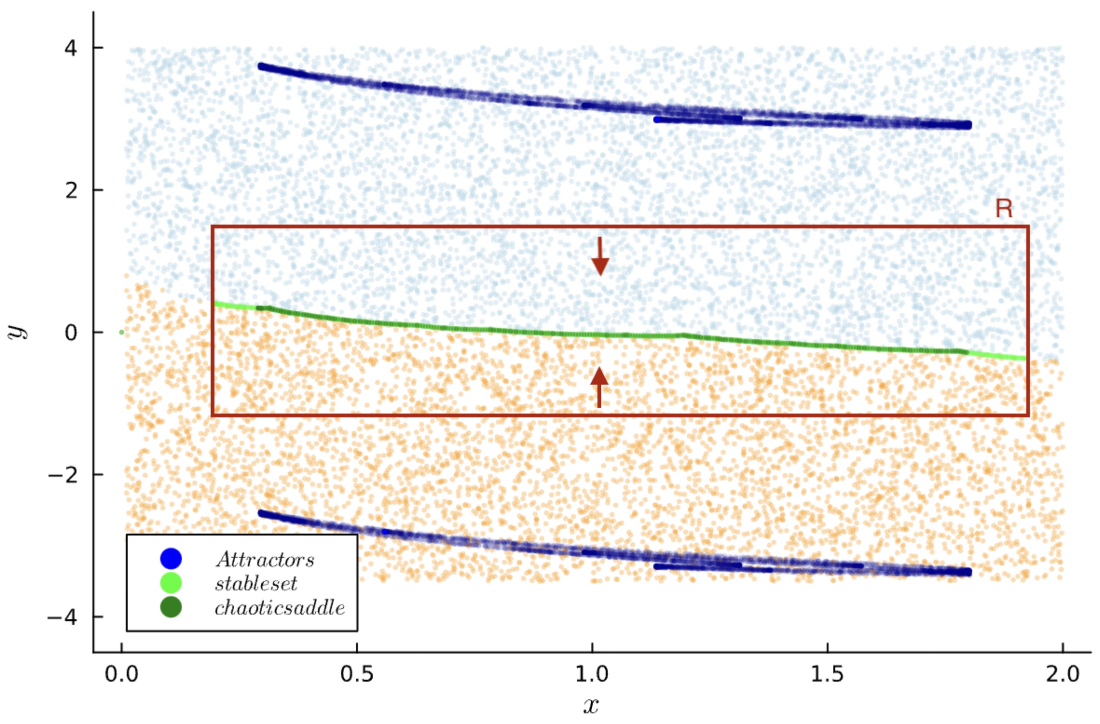
Numerical Example

We illustrate our findings using a simple skew-product map as an example:

$$x_{n+1} = 4.2 x_n \exp(-x_n^2)$$
$$y_{n+1} = y_n + 0.4\sin(y_n) + 0.4(x_n^2 - 1.4)$$

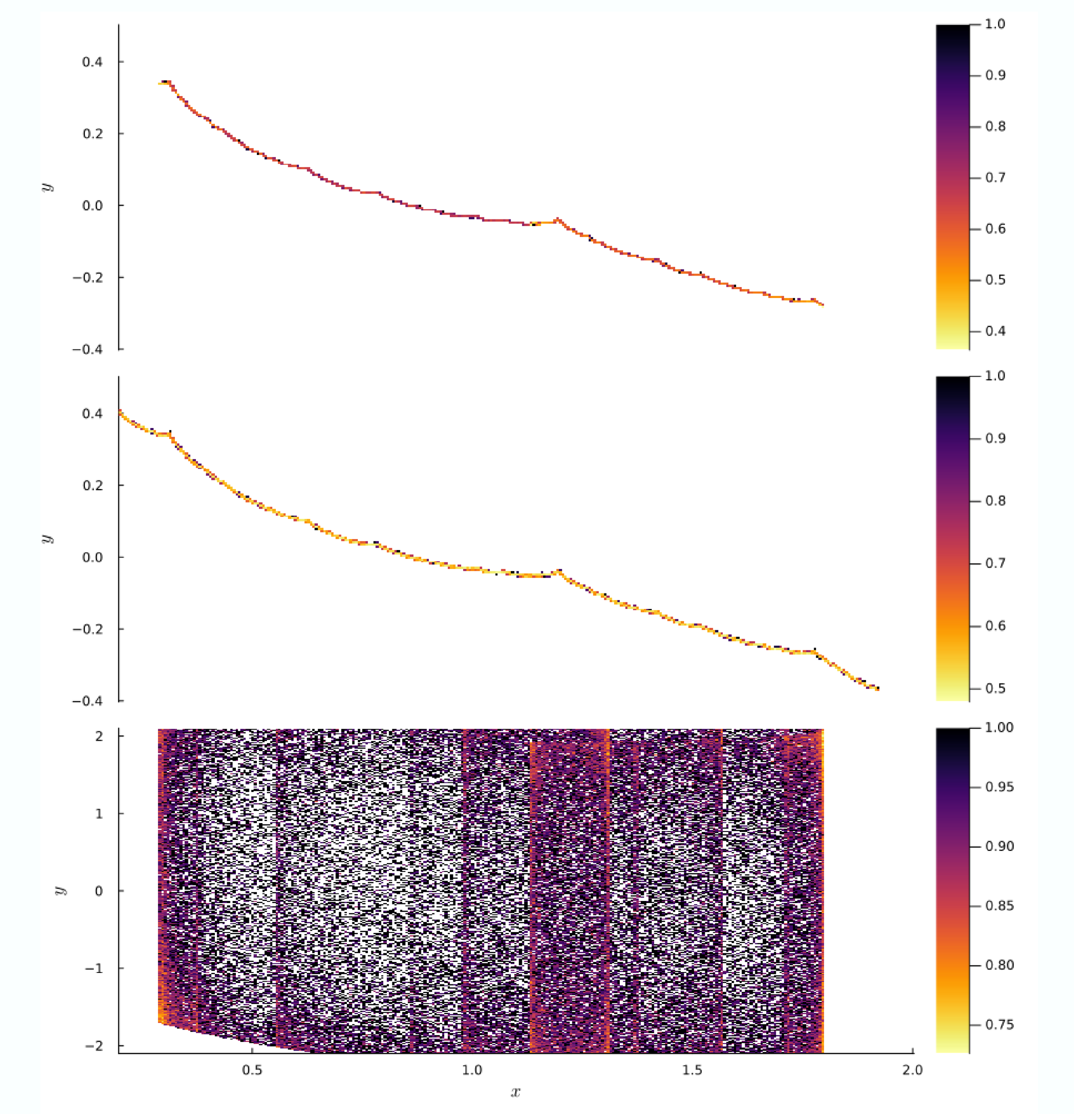
with $\{x_n\}, \{y_n\} \in \mathbb{R}$ and $n \in \mathbb{N}$.

The system shows chaotic behaviour in the x-direction and is multistable in the y-direction.



Plot of the setup of the system.

The following figure shows the numerical approximation of formula (6) with m chosen such that we get a measure on the saddle, on its stable or on its unstable set.



Approximation of: **top:** *the invariant probability measure on the saddle,* **middle:** *a measure on the stable set,* **bottom:** *a measure on the unstable set*

Fractal Dimension

1. Computing the box-counting dimension of the saddle gives ≈ 1.04.
2. Using a formula from [1] to compute the dimension of the saddle from the Lyapunov exponents gives ≈ 1.04.

Measure on the stable and unstable set

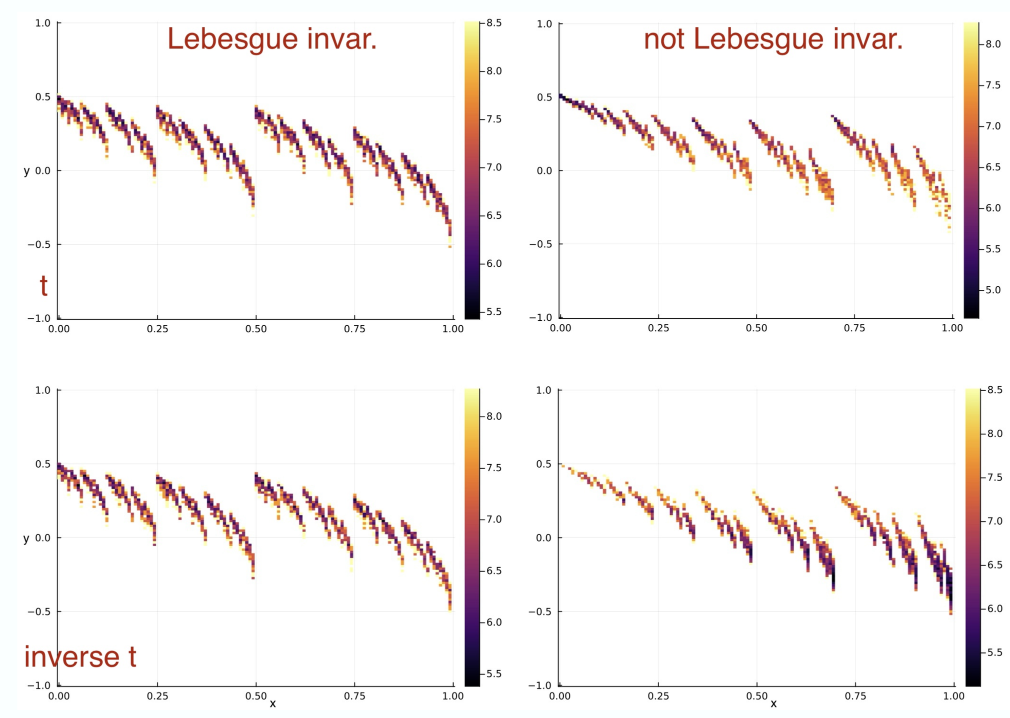
Setting $m(n) = 0$ gives a measure on the stable set, and setting $m(n) = n$ gives a measure on the unstable set. However, these measures are not invariant probability measures.

Additionally, there is a problem with this definition: We would expect that under time inversion, the stable set of the saddle becomes the new unstable set and vice-versa. For the measures on these sets, we would require an analogous behaviour. We can show that this expected relationship between the measure on the stable and unstable set holds if the dynamics are ℓ-invariant. If the system is not ℓ-invariant, the desired relation might be violated as the following example shows.

We define the skewed Bakers Map as:

$$S(x,y,z) = \begin{cases} (\frac{x}{q}, ys, z + 0.4\sin(z) + 0.4(x^2 - 1.4)) & , x \leq q \\ (\frac{x-q}{1-q}, \frac{1-s}{y} + s, z + 0.4\sin(z) + 0.4(x^2 - 1.4)) & , q < x \end{cases}$$

with $x \in [0,1]$. The measures on the stable and unstable set are shown in the next figure, and we can see that the mass of the measures is distributed differently in the case of the system that is not ℓ-invariant.



first row: Measure on the *stable* set in forward time, **second row:** Measure on the *unstable* set in backward time, **first column:** $q = s = 1/2$, thus the system is ℓ-invariant, **second column:** $q = 0.7, s = 0.51$, thus the system is not ℓ-invariant.

Continuous Systems

In continuous systems, we can define a measure on a non-attracting invariant set in a similar way: We define the two sets that we considered before for maps analogously:

$$R^{(t)} = \{x \in R : \varphi(\tau, x) \in R, \tau \in [0,t]\}$$

$$R^{(m,t,C)} = \{x \in R : \varphi(\tau, x) \in R, \tau \in [0,t] \;\&\; \varphi(m,x) \in C\}$$
$$= R^{(t)} \cap \varphi(-m, C)$$

with $m \leq t$. Now, we can first define the pre-measure μ on open sets $C \subset R$ by

$$\mu(C) = \lim_{t\to\infty} \frac{l(R^{(m(t),t,C)})}{l(R^{(t)})}, \tag{13}$$

and extend it to all Borel sets using Caratheodory's extension theorem as before. The mediating time $m(t)$ is again such that $0 \ll m(t) \ll t$ as $t \to \infty$.

Open Questions

1. On the (un)stable set, we can define an invariant measure that has infinite measure but not an invariant probability measure. How can we use this in applications?
2. The formula from [1] for the Lyapunov dimension seems to work only for typical cases, what can we do in atypical cases?
3. How can we compute and use the defined measures in higher dimensions to be useful for applications like modelling climate tipping points?
4. In the case of continuous time, under certain conditions, we can probably use algorithms like Edge-Tracking or PIM-triple to sample the measure. What are these conditions and what are the convergence properties?

References

[1] D. Sweet, Edward Ott. Fractal Dimension of Higher-Dimensional Chaotic Repellors. *Physica D: Nonlinear Phenomena*, 139(1):1–27, 2000.

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