EARLY-WARNING SIGNS FOR SPDES WITH BOUNDARY NOISE

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Introduction

This poster discusses the effect of boundary Gaussian additive noise on multistable partial differential equations (PDEs). In particular, such a term is involved in constructing early-warning signs (EWSs) able to predict the crossing of deterministic bifurcations thresholds. The analytic results are applied to a two-dimensional ocean model.

Description of the studied systems

We introduce the fast-slow system perturbed by white noise $\dot{W}$,

$$\begin{cases} \mathrm{d}u(x,t) = (F_1(p(x,t))\, u(x,t) + F_2(u(x,t),p(x,t)))\, \mathrm{d}t & \text{for } x \in \mathcal{X}, \\ \gamma(p(x,t))\, u(x,t) = \sigma B \dot{W}(x,t) & \text{for } x \in \partial\mathcal{X}, \\ \mathrm{d}p(x,t) = \epsilon G(u(x,t),p(x,t))\, \mathrm{d}t & \text{for } x \in \mathcal{X}\ . \end{cases} \quad (1)$$

We observe the linearized fast system, with $\epsilon = 0$, on a steady solution $u_*^{(p)}$, thus obtaining

$$\begin{cases} \mathrm{d}u(x,t) = A(p)\, u(x,t)\mathrm{d}t & \text{for } x \in \mathcal{X}, \\ \gamma(p)\, u(x,t) = B\dot{W}(x,t) & \text{for } x \in \partial\mathcal{X}, \\ u(x,0) = u_0(x) & \text{for } x \in \mathcal{X}\ . \end{cases} \quad (2)$$

The time autocovariance of the solution of (1) with an initial condition close to $u_*^{(p)}$ and the time autocovariance from (2) with u_0 in a neighbourhood of the null function are expected to be similar under small noise perturbations and long times. We assume the linear operator A_0 such that

$$A_0(p)v = A(p)v \quad \text{for any } v \in \mathcal{D}(A_0(p)) = \mathcal{D}(A(p)) \cap \mathcal{D}(\gamma(p)) \cap \{\gamma(p)\, v(x) = 0 \text{ for } x \in \partial\mathcal{X}\}\ ,$$

to be negative solely for $p < \lambda \in \mathbb{R}$ and non-positive for $p = \lambda$. We assume that there exists a constant $c(p) \in \mathbb{R}$ such that $(A_0(p) + c(p))^{-1}$, for $p \le \lambda$, is compact. This entails that the spectrum of $A_0(p)$ is discrete and labeled as $\left\{\lambda_i^{(p)}\right\}_{i\in\mathbb{N}_{>0}}$. The generalized eigenfunctions of $A_0(p)^*$ corresponding to $\left\{\overline{\lambda_i^{(p)}}\right\}_{i\in\mathbb{N}_{>0}}$ are labeled as $\left\{e_{i,k}^{(p)*}\right\}_{i\in\mathbb{N}_{>0}}$, for k their rank, and assumed to be continuous on $p \le \lambda$ in $L^2(\mathcal{X})$. We introduce the time-asymptotic autocovariance as

$$V_\infty^{\tau} := \lim_{t_2\to\infty} V_{(t_1,t_2)}\ ,$$

for fixed $\tau = t_1 - t_2$ and $V_{(t_1,t_2)}$ that satisfies

$$\left\langle v, V_{(t_1,t_2)} w \right\rangle = \mathrm{Cov}(\langle u(\cdot,t_1), v\rangle\, , \langle u(\cdot,t_2), w\rangle)\ .$$

Theorem (Construction of EWSs)

We set $\tau \ge 0$. Under non-restrictive assumptions on γ and B, we assume that the generalized eigenfunctions of $A_0(p)^*$ are complete in $L^2(\mathcal{X})$ for any $p \le \lambda$.

a) *We set the sequences $\left\{f_1^{(p)}\right\}, \left\{f_2^{(p)}\right\}$ continuous in $L^2(\mathcal{X})$ for $p \le \lambda$. Then for any $\delta > 0$ there exist two sequences $\left\{g_1^{(p)}\right\}, \left\{g_2^{(p)}\right\}$ continuous in $L^2(\mathcal{X})$ such that $g_1^{(p)}, g_2^{(p)} \in \mathcal{D}(A_0(p)^*)$,*

$$\left|\left|f_1^{(p)} - g_1^{(p)}\right|\right| < \delta \quad , \quad \left|\left|f_2^{(p)} - g_2^{(p)}\right|\right| < \delta\ ,$$

for any $p \le \lambda$, and

$$\left|\left\langle g_1^{(p)}, V_\infty^{\tau} g_2^{(p)}\right\rangle\right| = \Theta\left(-\operatorname{Re}\left(\lambda_1^{(p)}\right)^{-2M_1+1}\right) \quad (3)$$

for $p \to \lambda^-$ and M_1 the dimension of the generalized eigenspace of $A_0(p)^*$ corresponding to $\overline{\lambda_1^{(p)}}$.

b) *We set $p < \lambda$. The time-asymptotic autocorrelation nonlinear operator of lag time τ, labeled $\hat{V}_\infty^{\tau}$ and defined as*

$$\hat{V}_\infty^{\tau}(v,w) = \frac{\langle v, V_\infty^{\tau} w\rangle}{\langle v, V_\infty^{0} w\rangle}$$

for any $v, w \in \mathcal{D}(A_0(p)^*)$ such that $\langle v, V_\infty^0 w\rangle \neq 0$, satisfies

$$\hat{V}_\infty^{\tau}\left(e_{i,1}^{(p)*}, f\right) = e^{\overline{\lambda_i^{(p)}}\tau} \quad (4)$$

for any $i \in \mathbb{N}_{>0}$ and f in a dense subset $\mathcal{H}'$ of $L^2(\mathcal{X})$ such that $\left\langle e_{i,1}^{(p)*}, V_\infty^0 f\right\rangle \neq 0$.

A two-dimensional Boussinesq model

The Boussinesq model, studied in [1], describes different properties of a two-dimensional region of the ocean. Such area is defined by the spatial variables $(z,x) \in [-H,0] \times [0,L]$ for depth H and latitude length L. The scaled and non-dimensionalized variables that define the model are the salinity S, the temperature T, the vorticity ω and the streamfunction ψ. The two-dimensional system is described as follows,

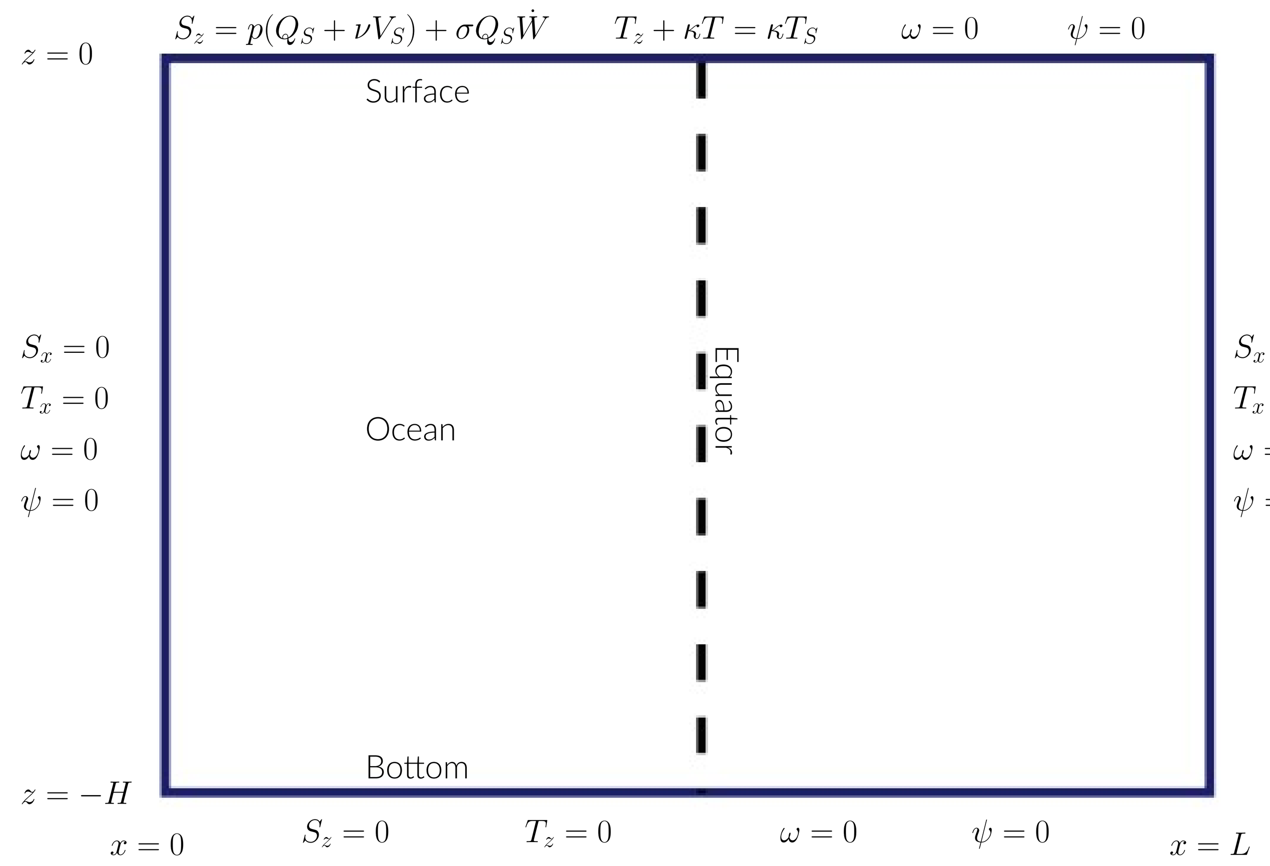
$$Pr^{-1}\left(\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x} + w\frac{\partial\omega}{\partial z}\right) = \Delta\omega + Ra\left(\frac{\partial T}{\partial x} - \frac{\partial S}{\partial x}\right)\ ,$$

$$\omega = -\Delta\psi \quad , \quad u = \frac{\partial\psi}{\partial z} \quad , \quad w = -\frac{\partial\psi}{\partial x}\ ,$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \Delta T\ ,$$

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + w\frac{\partial S}{\partial z} = Le^{-1}\Delta S\ ,$$

with boundary conditions displayed in the subsequent figure.



Aside from p, the parameters are fixed. The functions Q_S and T_S on x are assumed to be symmetric on the equator, and the function V_S endorses asymmetry in the system.

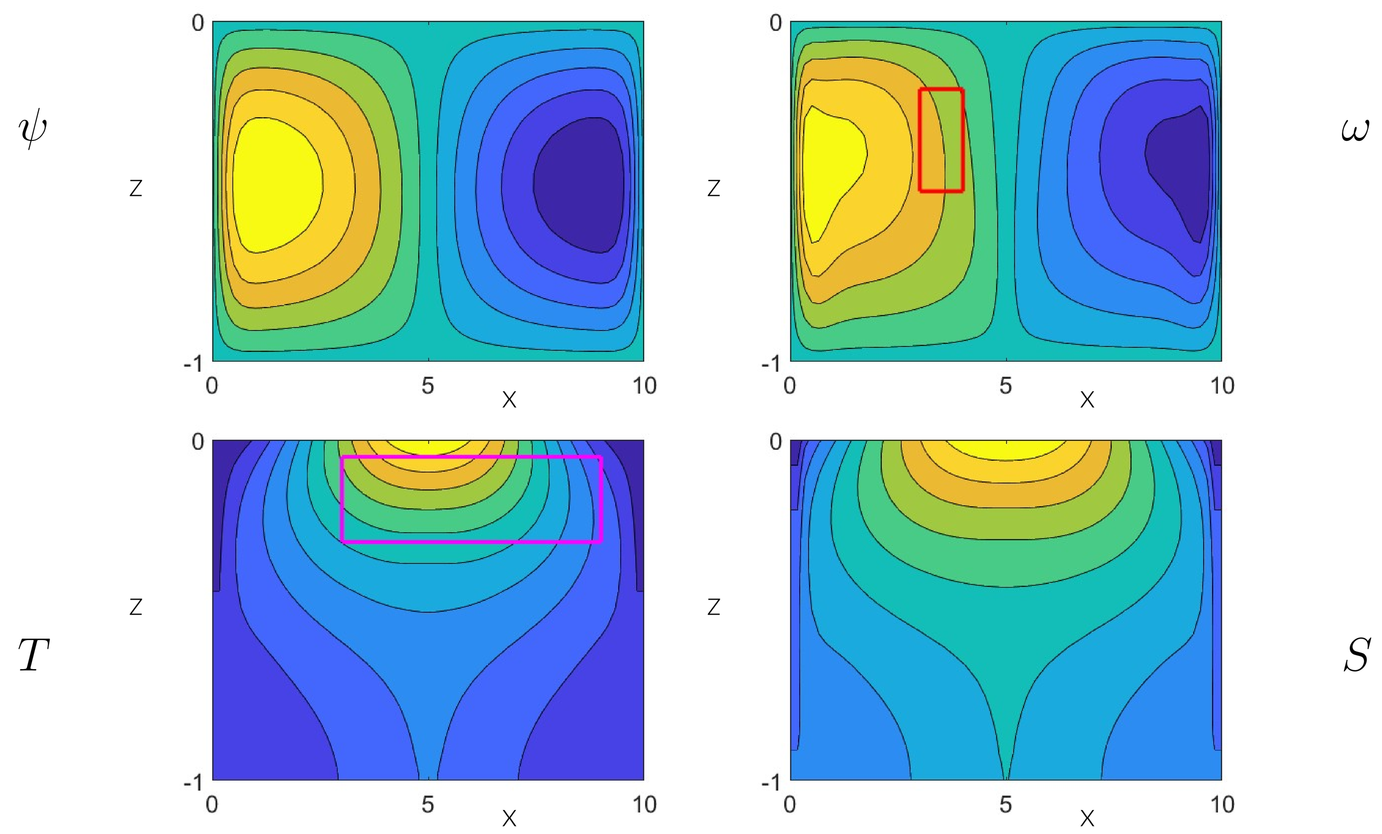


Fig. Components of a stable solution u_* for fixed p. The red rectangle displays the boundary of the support of the indicator function g_1 and the magenta rectangle delimits the support of the indicator function g_2.

References

[1] Henk A Dijkstra and M Jeroen Molemaker. Symmetry breaking and overturning oscillations in thermohaline-driven flows. *J. Fluid Mech.*, 331:169–198, 1997.

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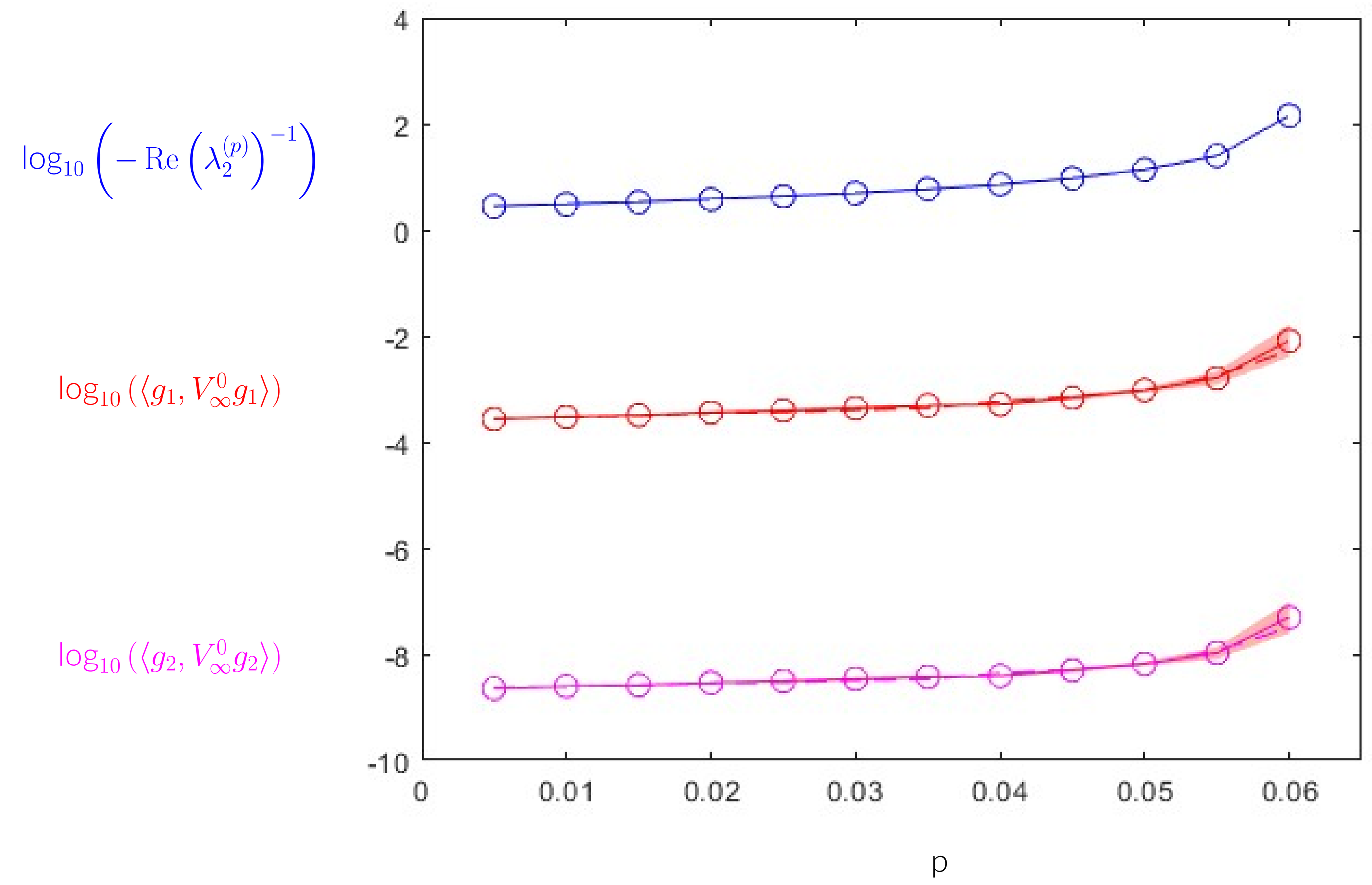
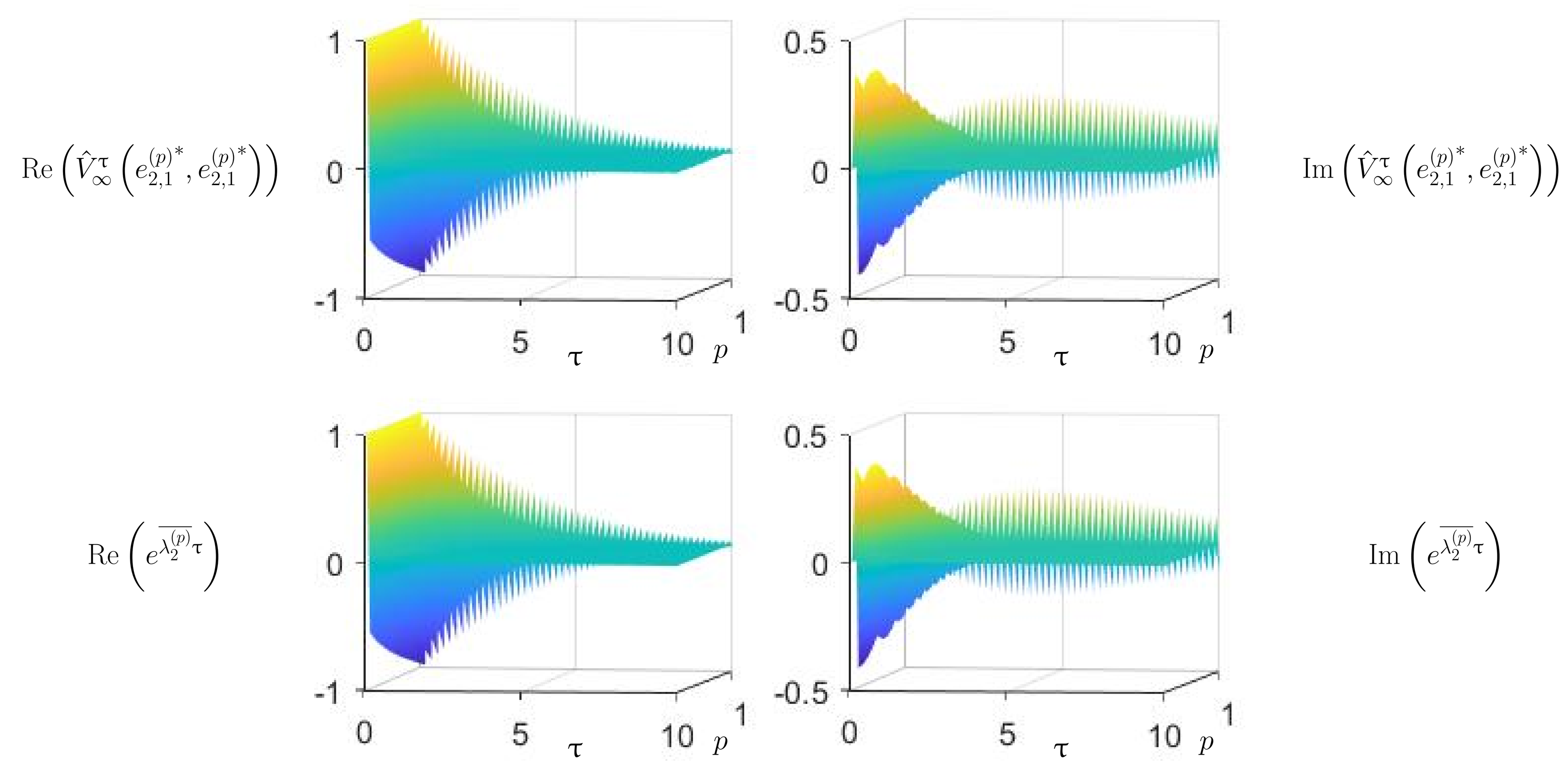
Application of the EWSs

To apply the early-warning signs on the Boussinesq model, we observe for a long time the time autocovariance of a translation of its solution, such that the boundary conditions in γ are homogenized. The rate in (3) is observed in the figure to follow, under the consideration that g_1 and g_2 are orthogonal to the generalized eigenspace of $A_0(p)$ corresponding to λ_1. Also, the generalized eigenspace of $A_0(p)^*$ related to $\overline{\lambda_2^{(p)}}$ has dimension equal to 1 for any p. In the case displayed below, the sign predicts the crossing of a supercritical pitchfork bifurcation threshold $\lambda \approx 0,063$.



The time autocorrelation of a solution of the system under different parameters is observed for a long time in the figure below. The sign anticipates the approach to a saddle-node bifurcation threshold $\lambda \approx 1$. The integral of the differences of the real and imaginary parts of the numerically studied quantities appears to be of order 10^{-5}. Such values are expected to be small from (4).



Conclusion

We have constructed two early-warning signs in the form of the qualitative behaviour of time-asymptotic autocovariance and quantitative growth of the time-asymptotic autocorrelation of the solution of a linearized system. The numerical application of the results is shown to be possible under the required considerations. Such an outcome expands the theory of early-warning signs ([2]) on the field of stochastic partial differential equations under boundary noise.

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