Extreme value methods in dynamical systems of different complexity

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Presenting:
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Abstract:

Recently an Extreme Value Theory based algorithm has found a plethora of applications in the field of climate dynamics. This algorithm allows us to study local properties of attractors of chaotic systems and gives information about the predictability and evolution of extreme events. However, this method requires subtle mathematical properties that are unlikely present in a system of such complexity. Here we try to give a solid mathematical foundation to this method and discuss the kind of phenomena that arise as we increase the complexity of the systems we study.

1. The algorithm

Description

This method allows to analyse a trajectory of a dynamical system and obtain information about the geometry of the attractor at a local scale, which in turn is related to the persistence of particular events and their predictability. For some examples of climate applications see References below.

The method works as follows: take a trajectory of the system and compute pairwise distances. Define a random variable whose extreme events are close recurrences to ball centred in a reference point in the trajectory and set a high threshold. Then take the exceedances over the threshold and fit a Generalized Pareto Distribution (GPD) to them.

Algorithm 1 X is a trajectory

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for x_i ∈ X do
    Observations_i = [−ln(||x_j − x_i||) for x_j ∈ X]
    ExtremeValues_i = {y ∈ Observations_i | y > threshold}
    σ_i = mean(ExtremeValues_i − threshold)
    LocalDimension_i = 1/σ_i
end for
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Mathematical (loose) justification

Define a random process $X_n = -\log(||T^n(x_0) - \zeta||)$.

- Set a high threshold t and define the *excess distribution function*:

$$\mathbb{P}(X_n - t \le x | x > t) = 1 - \frac{\mu(B_{e^{-x-t}}(\zeta))}{\mu(B_{e^{-t}}(\zeta))}$$

- When t increases, it converges to a Generalized Pareto Distribution (GPD)

$$GPD(x;\xi,\mu,\sigma) = \begin{cases} 1 - \left(1 + \frac{\xi(x-\mu)}{\sigma}\right)^{-1/\xi} & \text{if } \xi \neq 0 \\ 1 - e^{-(x-\mu)/\sigma} & \text{if } \xi = 0 \end{cases}$$

- Assume regular variation of the tails, we obtain

$$1 - \frac{\mu(B_{e^{-x-t}}(\zeta))}{\mu(B_{e^{-t}}(\zeta))} \simeq 1 - \frac{l(e^{-x-t})e^{-x-td_\zeta}}{l(e^{-t})e^{-td_\zeta}} \simeq 1 - \left(\frac{e^{-x-t}}{e^{-t}}\right)^{d_\zeta} = 1 - e^{-d_\zeta x} = GPD_\zeta(x;0,0,1/d_\zeta)$$

where

$$d_\zeta = \dim_{loc}(\zeta) = \lim_{r\to 0} \frac{\log \mu(B_r(\zeta))}{\log r} \simeq \frac{1}{\sigma_\zeta}$$

2. The easiest case: absolutely continuous μ

Theorem: For ergodic systems whose invariant measure μ is absolutely continuous w.r. Lebesgue measure it can be proven that:

i) Exceedances converge to a GPD distribution with parameter $\sigma_\zeta = \frac{1}{d_\zeta}$ for a.e. initial condition x_0 and a generic ζ.

ii) If the density is analytical in a neighbourhood of ζ, the average of the exceedances converges to the parameter sigma with error bounds $o(n^{-1/2})$ (if strong mixing) on the number of exceedances n and $o(e^{-t})$ on the threshold t for large n.

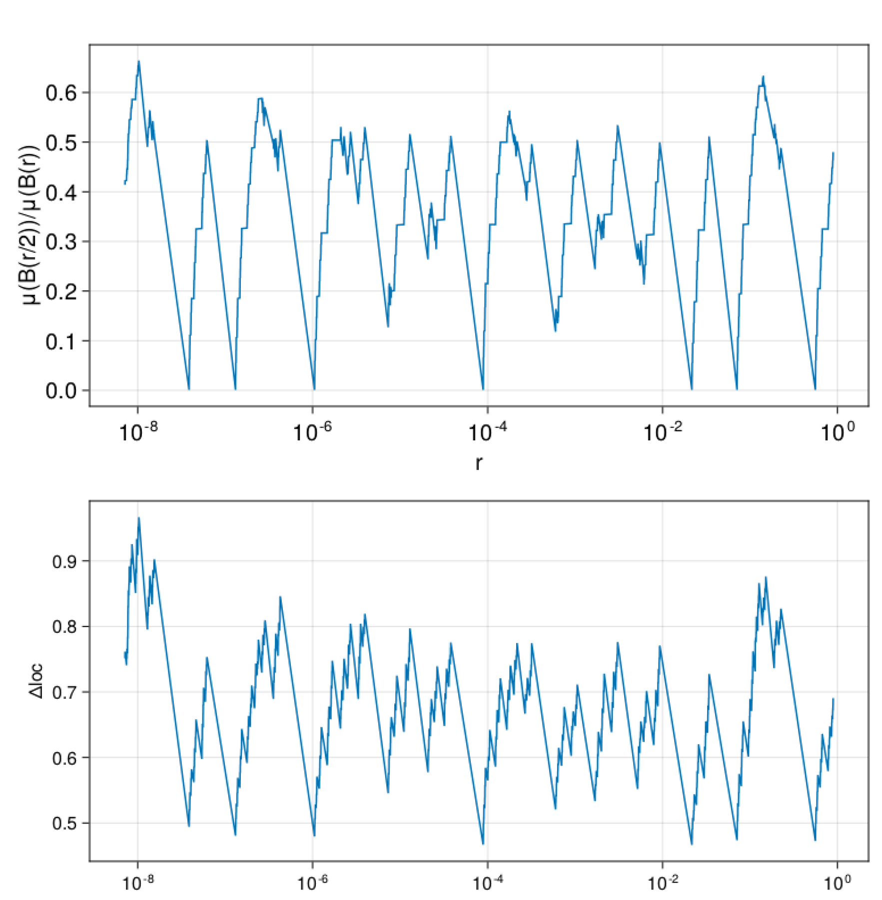
3. 1D discontinuities and special points

For the following special points, the consequences of theorem can be shown to hold despite not meeting the hypothesis:

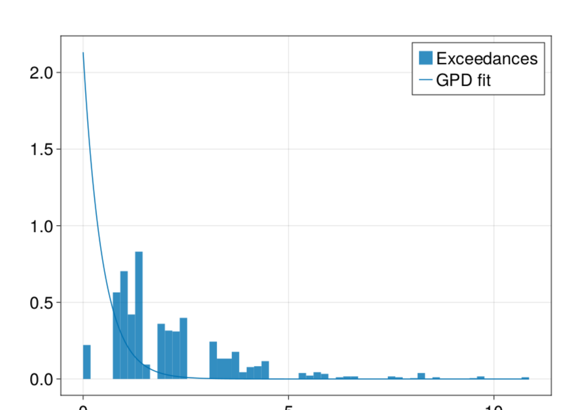
- Jump discontinuity points of the measure ($\zeta = 1/\beta$ in the beta transformation when beta is the golden ratio)
- Integrable poles ($\zeta \in \{0,1\}$ in the logistic map)
- Indifferent points where the density exists and is finite ($\zeta = -1$ in the Cusp map)
- Points where the density function is 0 ($\zeta = 1$ in the Cusp map)

4. 1D Singular measure: The Cantor Shift

Any measure that takes support on a Cantor set is singular w.r. Lebesgue measure. This has an effect in the regularly varying properties of the measures on it since shrinking the ball in a gap does not change its measure. Hence the algorithm does not work:



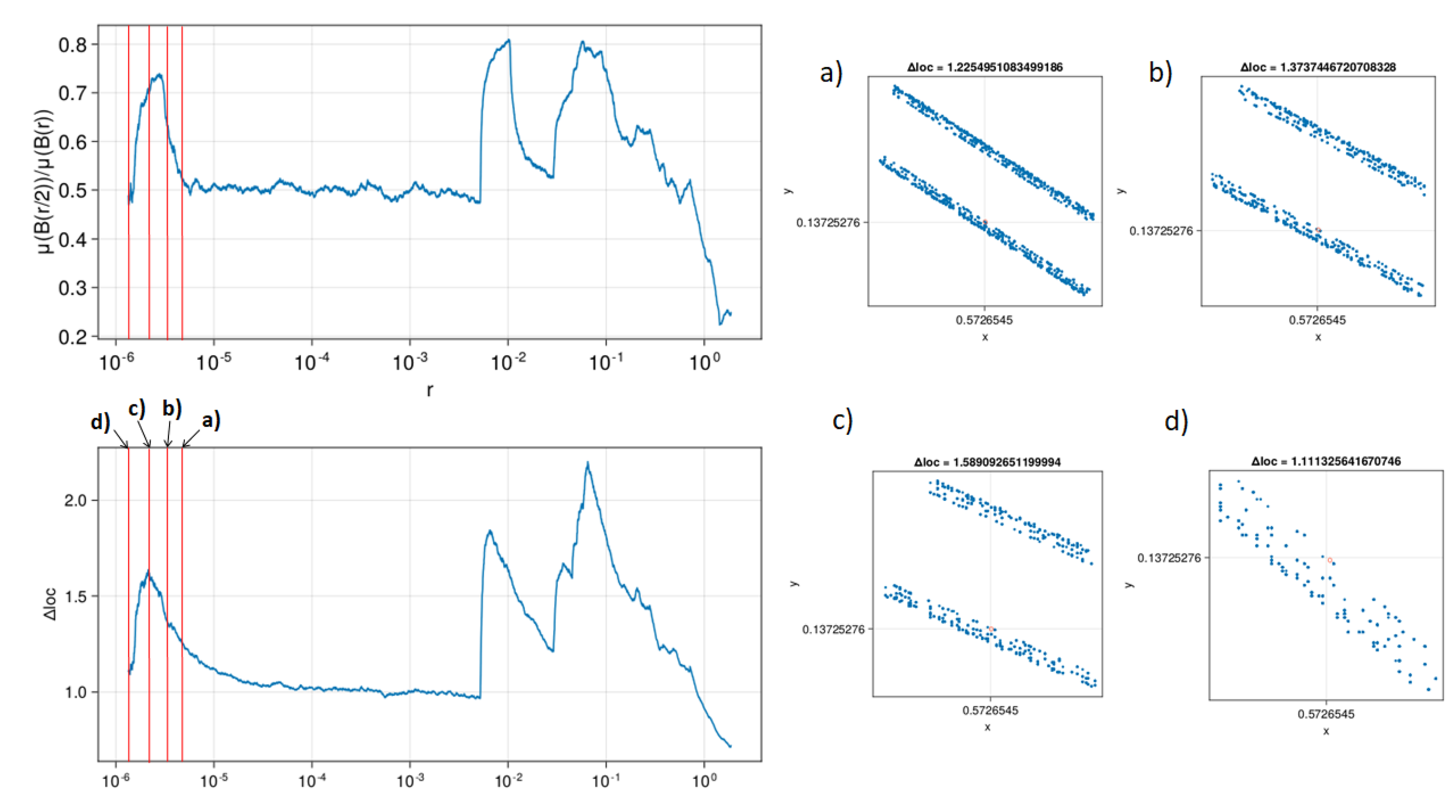
Numerical estimation of the regularly varying properties and the local dimension



Exceedances and GPD fit

5. General hyperbolic maps

For general hyperbolic maps, the invariant measure on the attractor is general singular w.r. Lebesgue, and can be extremely irregular. There are gaps, like in the Cantor Shift, that affect the regular variation properties and the local dimension estimation. This makes the estimate scale dependant, which means that depends in the length of the trajectory, which in applications is arbitrary.



Numerical estimation of the regularly varying properties and the local dimension for point in the attractor of the Hénon map.

References

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