EGU25-2192

Random number generation in kinetic plasma simulation: flattop and gamma distributions

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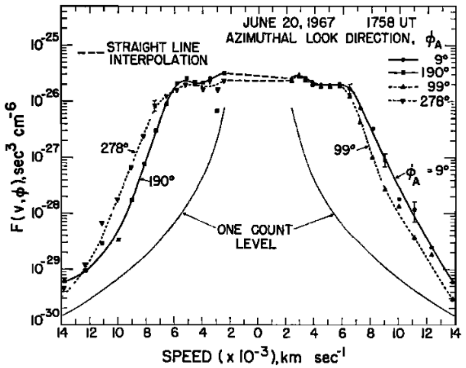
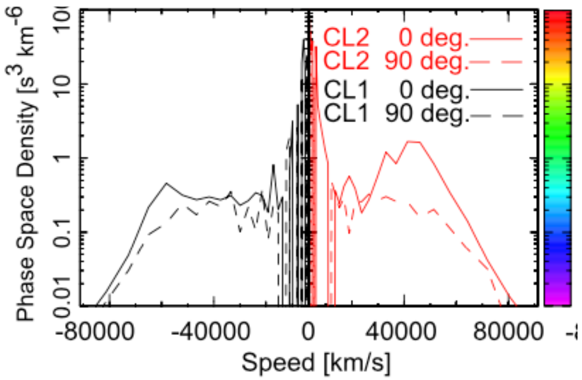
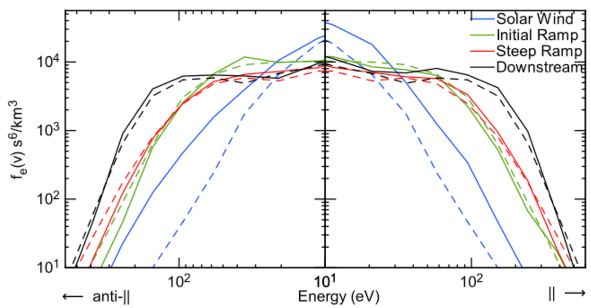
Flattop distribution



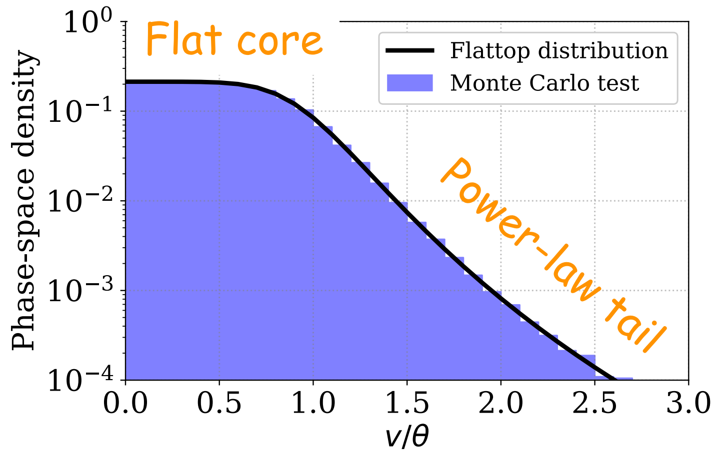
Montgomery+ 1970 JGR



Asano+ 2008 JGR



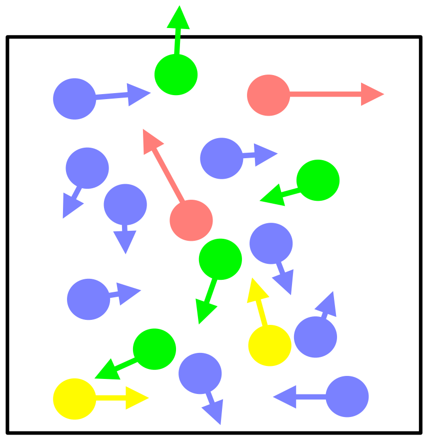
Schwartz+ 2011 PRL

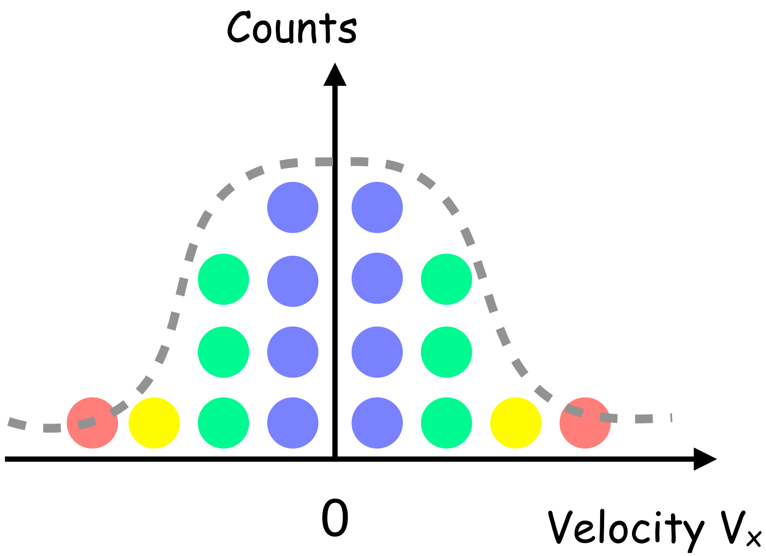


- Popular power-law form (Thomsen+ 1983, Oka+ 2022)

$$f_{\rm ft}(\boldsymbol{v})d^3v \sim \left(1+\left(\frac{v_\parallel^2}{\theta_\parallel^2}+\frac{v_\perp^2}{\theta_\perp^2}\right)^{\kappa}\right)^{-\frac{\kappa+1}{\kappa}} d^3v$$

Particle-in-Cell (PIC) simulation

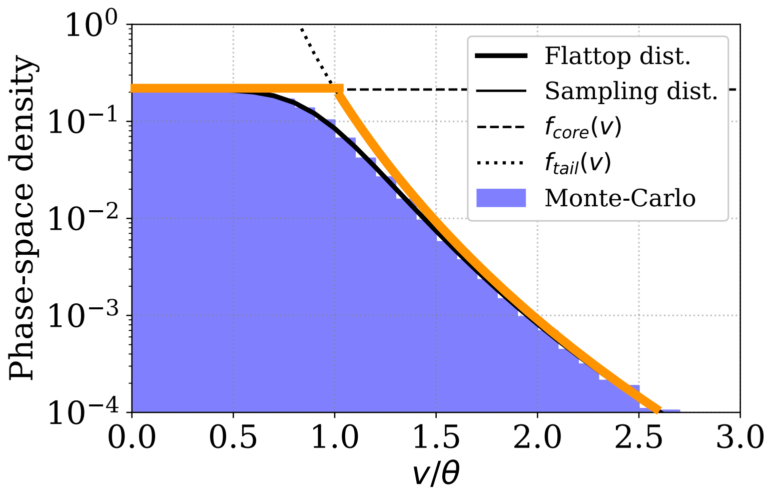




How to generate a flattop distribution by using random numbers?

- Maxwell distribution - well known
- Kappa distributions - known (Abdul & Mase 2015 PoP, SZ & Nakano 2022 PoP)
- Flattop distribution - unknown

(A) Piecewise rejection method



- Envelope (flat core + power-law tail)
- Acceptance efficiency: 60%~

$$\mathrm{Eff}_{\mathrm{ft}} = \frac{\Gamma(1+3/2\kappa)\Gamma(2-1/2\kappa)}{\Gamma(2+1/\kappa)}$$

Gamma function

$p_1 \leftarrow \frac{2\kappa-1}{2\kappa+2}$, $p_2 \leftarrow \frac{3}{2\kappa+2}$

repeat

  generate $X_1, X_2 \sim U(0,1)$

  if $X_1 \le p_1$ **then**

    $x \leftarrow (X_1/p_1)^{1/3}$

    if $X_2 < (1+x^{2\kappa})^{-(\kappa+1)/\kappa}$, **break**

  else

    $x \leftarrow ((1-X_1)/p_2)^{1/(1-2\kappa)}$

    if $X_2 < (x^{-2\kappa}+1)^{-(\kappa+1)/\kappa}$, **break**

  endif

end repeat

generate $X_3, X_4 \sim U(0,1)$

$v_{\perp 1} \leftarrow 2\theta_\perp x \sqrt{X_3(1-X_3)}\cos(2\pi X_4)$

$v_{\perp 2} \leftarrow 2\theta_\perp x \sqrt{X_3(1-X_3)}\sin(2\pi X_4)$

$v_\parallel \leftarrow \theta_\parallel x\ (2X_3-1)$

return $v_{\perp 1}, v_{\perp 2}, v_\parallel$

(B) Beta-prime method

[Zenitani 2024, *Res. Notes of the AAS*]

- Generalized beta-prime distribution

$$\mathrm{B}'(x;\alpha,\beta,p,q) = \frac{\alpha p\,\Gamma(\alpha+\beta)}{q^{\alpha p}\Gamma(1+\alpha)\Gamma(\beta)} \left(1+\left(\frac{x}{q}\right)^{p}\right)^{-(\alpha+\beta)} x^{\alpha p-1}$$

- Flattop distribution in spherical coordinates

$$F_{\mathrm{ft}}(v) \equiv f_{\mathrm{ft}}(v) 4\pi v^2 = \frac{3N}{\theta^3} \frac{\Gamma\left(1+\frac{1}{\kappa}\right)}{\Gamma\left(1+\frac{3}{2\kappa}\right)\Gamma\left(1-\frac{1}{2\kappa}\right)} \left(1+\left(\frac{v}{\theta}\right)^{2\kappa}\right)^{-(\kappa+1)/\kappa} v^2$$

$$= NB'\left(v; \frac{3}{2\kappa}, 1-\frac{1}{2\kappa}, 2\kappa, \theta\right)$$

- Beta-prime random number

$$X_{\mathrm{B}'(\alpha,\beta,p,q)} = q\left(\frac{X_{\mathrm{Ga}(\alpha,\delta)}}{X_{\mathrm{Ga}(\beta,\delta)}}\right)^{1/p}$$

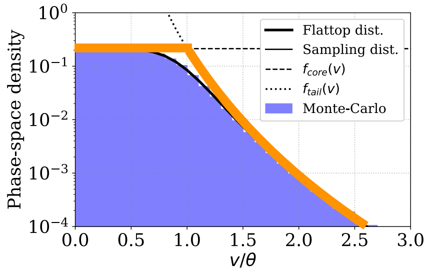
Beta-prime random number

Gamma random numbers

$$\begin{array}{l}
\text{generate } X_1 \sim \mathrm{Gamma}(3/(2\kappa), 1) \\
\text{generate } X_2 \sim \mathrm{Gamma}(1-1/(2\kappa), 1) \\
\text{generate } X_3, X_4 \sim U(0,1) \\
v \;\;\leftarrow (X_1/X_2)^{1/(2\kappa)} \\
v_{\parallel} \;\leftarrow \theta_{\parallel} v\ (2X_3-1) \\
v_{\perp 1} \leftarrow 2\theta_{\perp} v \sqrt{X_3(1-X_3)}\cos(2\pi X_4) \\
v_{\perp 2} \leftarrow 2\theta_{\perp} v \sqrt{X_3(1-X_3)}\sin(2\pi X_4)
\end{array}$$

Comparison

(A) Piecewise rejection



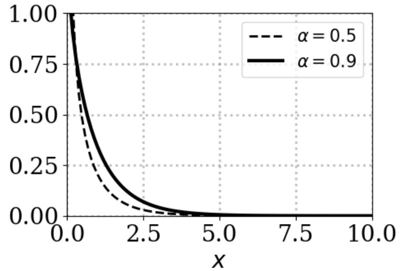
(B) Beta-prime method

$$X_{\mathrm{B}'(\alpha,\beta,p,q)} = q\left(\frac{X_{\mathrm{Ga}(\alpha,\delta)}}{X_{\mathrm{Ga}(\beta,\delta)}}\right)^{1/p}$$

Efficiency	60%~100%	100%*
# of random numbers	4	6
Effective # of random numbers	4.0~5.3	6.0~6.4
Other	Stand-alone	It requires Gamma generators

Gamma-distributed random number

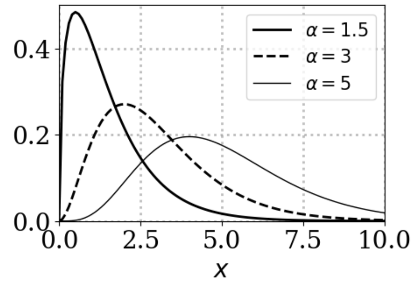
α<1

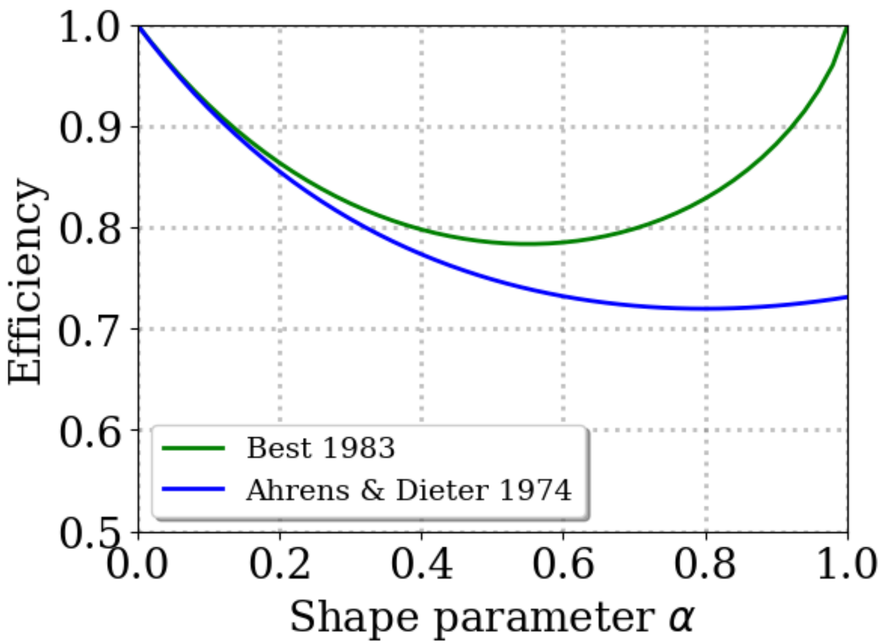


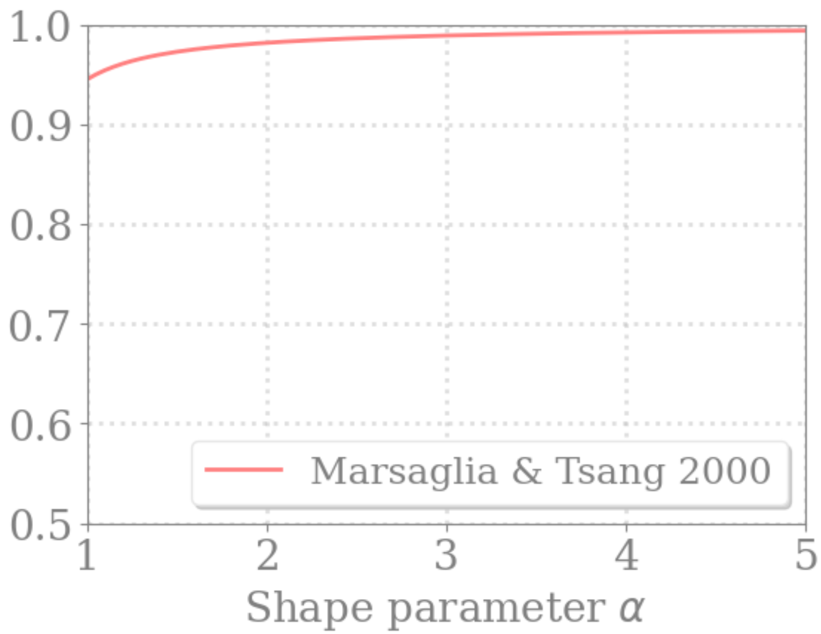
α: Shape parameter

$$\mathrm{Ga}(x;\alpha,1)=\frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}$$

1<α

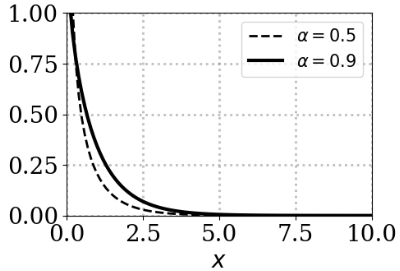






Gamma-distributed random number

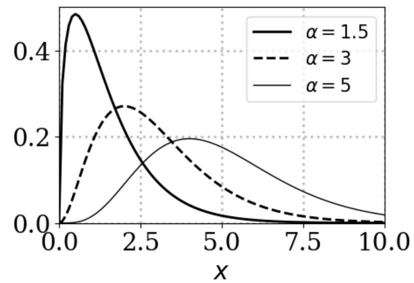
$\alpha < 1$



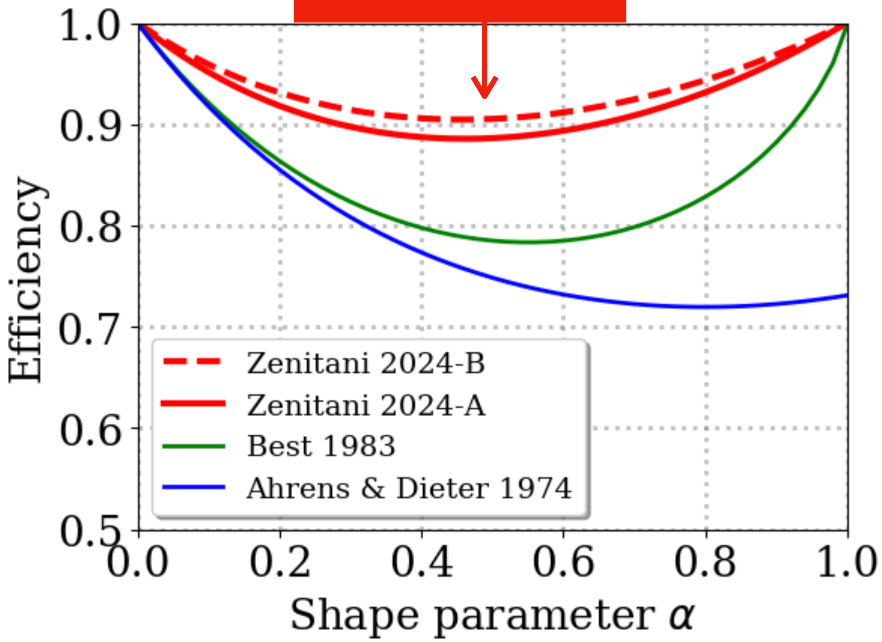
α: Shape parameter

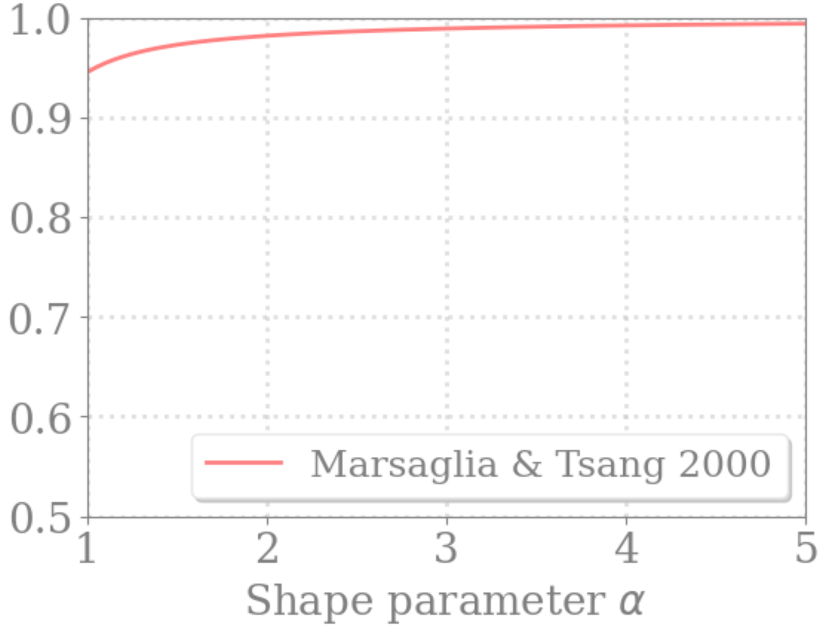
$$\mathrm{Ga}(x;\alpha,1) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}$$

$1 < \alpha$



Zenitani 2024





New gamma generator

[Zenitani 2024, *Economics Bulletin*]

- Generalized exponential function (Kundu & Gupta 2007)

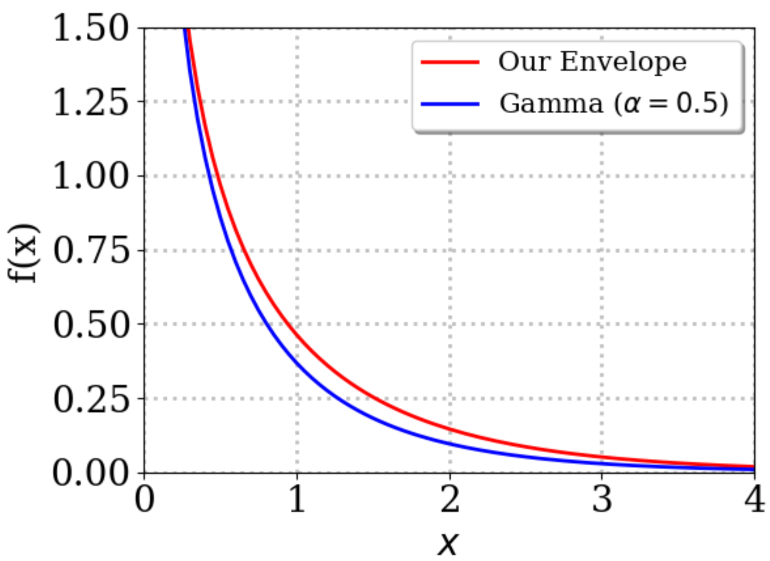
$$F_{\exp}(x;\alpha) = (1-e^{-x})^{\alpha}$$

- Its derivative

$$f_{\exp}(x;\alpha) = \alpha(1-e^{-x})^{\alpha-1}e^{-x}$$

Gamma distribution

$$f_{\Gamma}(x;\alpha) = \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}$$



Algorithm 1

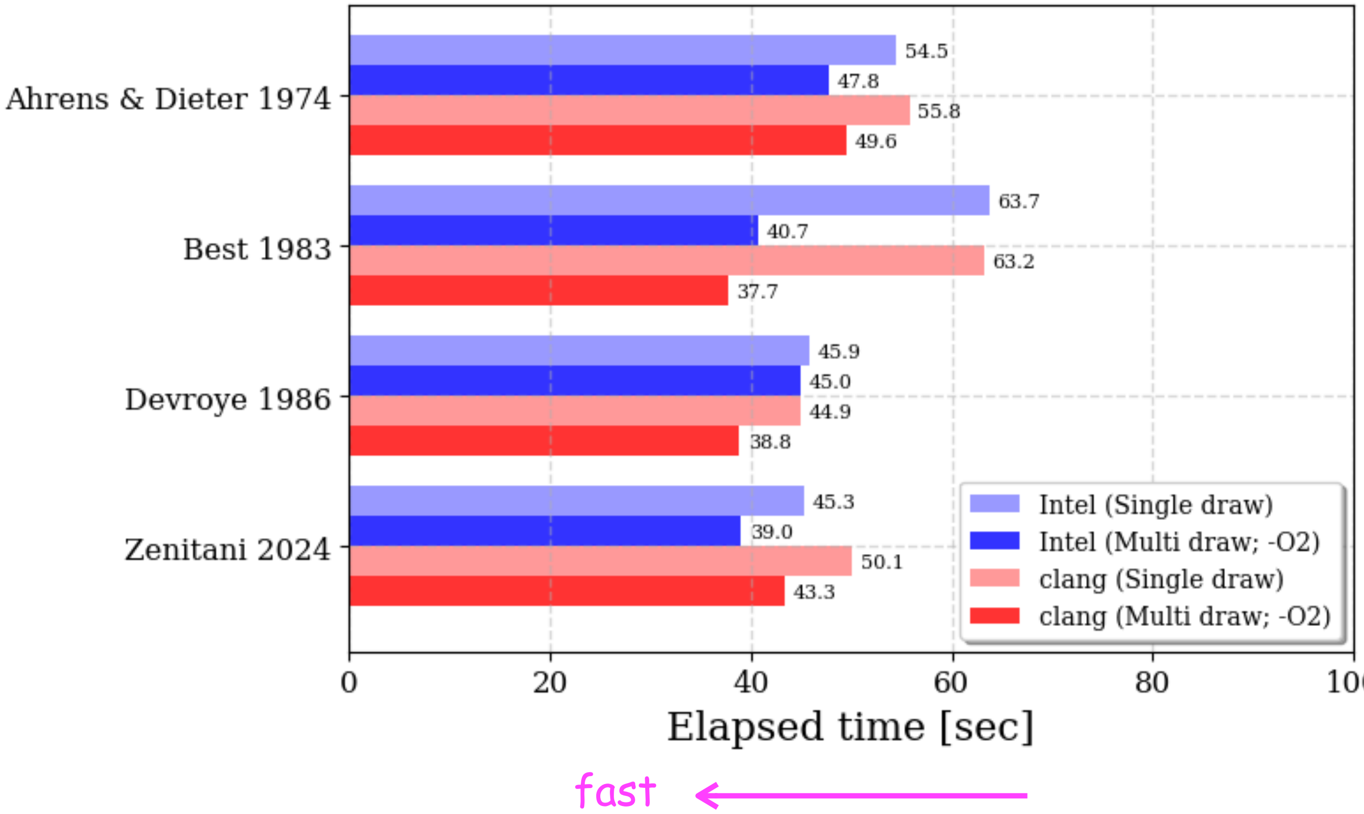
repeat
  generate $U_1, U_2 \sim U(0,1)$
  $b \leftarrow U_1^{1/\alpha}$, $x \leftarrow -\log(1-b)$
  if $U_2^{1/(1-\alpha)} x \le b$ **return** x
end repeat

Algorithm 2

repeat
  generate $U_1, U_2 \sim U(0,1)$
  $b \leftarrow U_1^{1/\alpha}$, $x \leftarrow -\log(1-b)$
  if $U_2(4+(1-\alpha)x) \le (4+(\alpha-1)x)$ **return** x
  if $U_2(4+(2-\alpha)x) \le (4+\alpha x)$ **then**
    if $U_2^{1/(1-\alpha)} x \le b$ **return** x
end repeat

New gamma generator

[Zenitani 2024, *Economics Bulletin*]



- Zenitani 2024 & Devroye 1986 (NumPy) are the best two

Summary

- Monte Carlo sampling of
  - Flattop distribution
    - (A) Piecewise rejection method
    - (B) Beta-prime method, with two gamma random numbers
  - Gamma distribution
    - A new gamma generator for a<1
    - One of best two, together with Devroye (1986)

- References:

  - [1] Zenitani, *Research Notes of the AAS*, **8**, 30 (2024) [Non-refereed]
  - [2] Zenitani, *Economics Bulletin*, **44**, 1113 (2024) [Refereed]