Simulation of Internal Waves within an ALE ocean model: numerical challenges and modelling

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Motivation: Internal Waves, Overturning, and Numerical Mixing I

Internal Solitary Waves (ISW) are a demanding benchmark because they combine strong stratification, nonhydrostatic dispersion, pycnocline deformation, and in the most difficult cases, overturning and breaking.

- In a fixed vertical grid, repeated remapping across the pycnocline creates **spurious diapycnal mixing** (SDM).
- In a purely Lagrangian / isopycnal description, SDM is reduced, but true overturning can be overly suppressed because vertical mass transfer is part of the resolved dynamics during breaking.
- The **ALE** strategy [5] offers the flexibility to design the vertical grid motion so that SDM is reduced without overly negating physically relevant vertical fluxes.

Motivation: Internal Waves, Overturning, and Numerical Mixing II

Aim: Reduce numerical mixing while retaining the vertical transfer needed for deforming and breaking waves.

SLS in Generalized Vertical Coordinates I

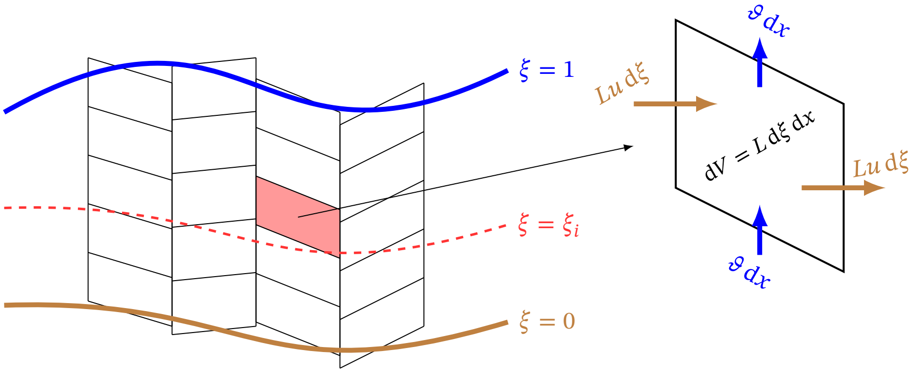
SLS is a hybrid Finite Volume (FV) / Finite Element (FE) solver for stratified, nonhydrostatic flows [3]. The vertical mesh is described by the map $z(x, y, \xi, t)$, where x, y are the horizontal coordinates, and ξ is the vertical parametric argument that labels the layers:

The local layer thickness is $L = \dfrac{\partial z}{\partial \xi}$, and the *Generalized Continuity Equation* states:

$$\underbrace{\frac{\partial L}{\partial t}}_{\text{volume change}} + \underbrace{\nabla_H \cdot (L\mathbf{u})}_{\text{Lagrangian impulse}} + \underbrace{\frac{\partial \vartheta}{\partial \xi}}_{\text{cross-layer coupling}} = 0$$

where $L\mathbf{u}$ is the horizontal volume flux through a moving layer, while ϑ is the **dia-surface ALE flux** and measures how much fluid crosses the moving layers.

SLS in Generalized Vertical Coordinates II



Sketch of the FV fluxes in SLS. The horizontal (Lagrangian) flux is $L\mathbf{u}$ and the vertical ALE flux is ϑ.

Why spurious diapycnal mixing appears I

For a passive density field with no physical diffusion ($\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$), the variance $\iiint \rho^2 V$ should be conserved. Any decay of the discrete variance is therefore a measure of numerical mixing [2]:

$$\frac{d}{dt} \iiint \rho^2 \, dV \sim - \iiint |\vartheta| \, \Delta z \, |\partial_z \rho|^2 \, dV \leq 0$$

So SDM increases when:

$|\vartheta|$ is large, Δz is coarse, $|\partial_z \rho|$ is large.

This gives the design principle for ALE movement: keep ϑ moderate, and concentrate resolution near the pycnocline where $|\partial_z \rho|$ is large.

Variational ALE: mesh update and cost functional I

The GCE separates the mesh evolution into horizontal Lagrangian transport and a cross-layer ALE correction. We write within the timestep $\Delta t = t^{n+1} - t^n$ the update as:

$$z^{n+1} = z^*_{lag} - \Delta t\, \vartheta$$

where z^*_{lag} denotes the mesh position predicted by the Lagrangian part of the GCE.
To design the ALE scheme, the ϑ in SLS is obtained by minimizing the functional [2]:

$$\mathcal{F}(\vartheta) = \iiint \Bigg[\underbrace{T_{ref}\, a_\vartheta \vartheta^2}_{\text{Lagrangian bias}} + \underbrace{\frac{1}{\Delta t} \left\| \boldsymbol{A}_S \nabla (z^{n+1} - z_{ref}) \right\|^2}_{\text{mesh regularity}} + \underbrace{\frac{a_M}{\Delta t} \left(M \frac{\partial z^{n+1}}{\partial \xi} \right)^2}_{\text{adaptive refinement}} \Bigg] dx\, d\xi$$

$$\boldsymbol{A}_S = \begin{bmatrix} \sqrt{a_x}\Delta x & & \\ & \sqrt{a_y}\Delta y & \\ & & \sqrt{a_\xi}\Delta \xi \end{bmatrix},$$

z_{ref}	:	Reference grid
$M(x, y, \xi, t)$	:	Monitor function
$T_{ref} \overset{\text{def}}{=} \sqrt{\frac{h\rho_0}{g(\rho_{max} - \rho_{min})}}$	:	Characteristic time scale

Variational ALE: mesh update and cost functional II

The parameters entering the functional are:

- a_ϑ: Controls Lagrangian bias.
- a_x, a_y, a_ξ: set the smoothing strength in the horizontal and vertical directions.
- a_M: weights the adaptive refinement term.

The monitor function is chosen so that mesh resolution is increased in areas with large density gradients: $M = |\partial_\xi \rho| / |\partial_\xi \rho|_{max}$

Elliptic problem for the ALE flux I

Based on the calculus of variations [4], the minimizer of $\mathcal{F}$ satisfies the following elliptic problem:

$$A\vartheta - \nabla \cdot (\boldsymbol{B}\nabla\vartheta) = R$$

where A, $\boldsymbol{B}$, R are coefficients that do not depend on ϑ.

ALE update within one timestep :

1. **Lagrangian step:** compute the Lagrangian GCE fluxes to get z^*_{lag}
2. **Elliptic problem:** solve for ϑ to minimize $\mathcal{F}$
3. **ALE update:** use this optimal ϑ in the vertical fluxes.

Test case I: ISW over a submerged wedge I

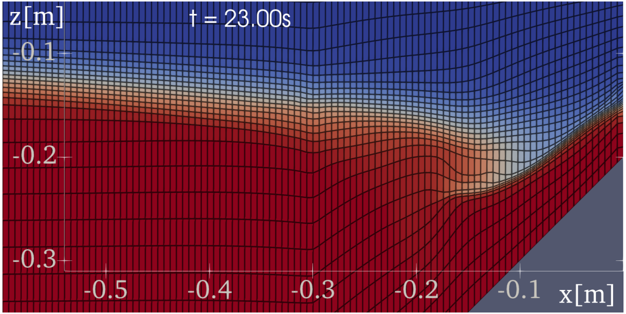
This test isolates the interaction between an ISW and steep topography. The wave remains coherent, the pycnocline undergoes strong local deformation, and we can compare with results from [6].

Setup

- ISW with thick pycnocline.
- Submerged wedge with a 45° slope.
- SLS simulations: ALE scheme with $a_\vartheta = 0.01\ ,\ \ a_x = a_\xi = 1\ ,\ \ a_M = 10$

What it tests

- Ability of the ALE mesh to concentrate near sharp density gradients.
- Suppression of SDM while maintaining good mesh quality.



ISW–wedge interaction: the ALE mesh remains concentrated around the deformed pycnocline while keeping the layers smooth

Test case II: Breaking-wave results and physical interpretation I

This test examines the performance in the breaking-wave regime, where strong nonlinear deformation and overturning occur. We adopt the configuration of [1, case 60] to study whether the ALE scheme can resolve the overturning with minimal SDM.

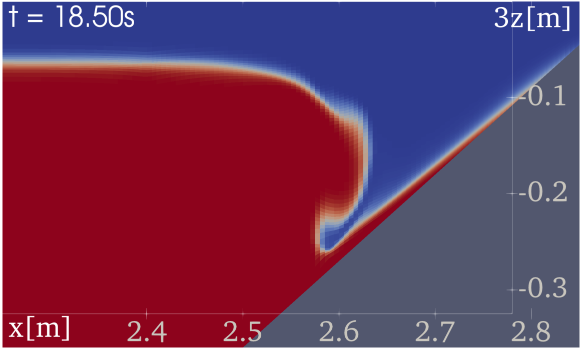
Setup

- An ISW propagating towards a sloping bottom
- Plunging breaking mechanism with overturning

Physics

- Induces significant dissipation
- Causes mixing through the pycnocline

Test case II: Breaking-wave results and physical interpretation II



(0.01, 10): balanced

Challenges

- Highly sensitive to SDM
- Exposes limitations of Lagrangian / Eulerian approaches

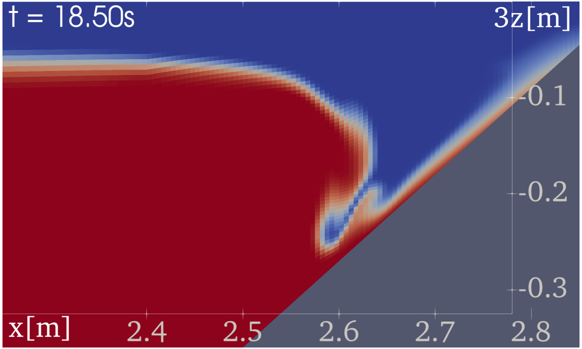
Parameter effect

- $a_x, a_\xi = 1$ fixed
- (a_ϑ, a_M) varying between runs

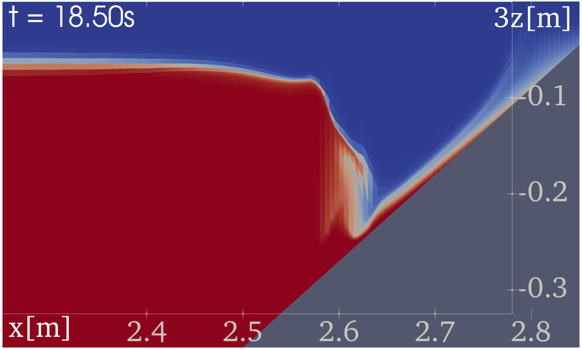
Takeaway

- **Moderate Lagrangian bias + monitor function refinement** gives best behaviour

Test case II: Breaking-wave results and physical interpretation III

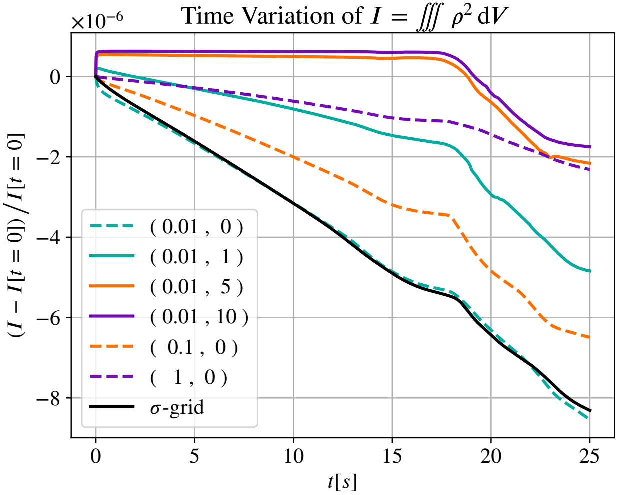


$(0.01, 0)$: overly diffusive



$(1, 0)$: overly constrained

Test case II: Effect on Spurious Diapycnal Mixing (SDM) I



Quantitative assessment

- Monitor-based refinement significantly reduces SDM up to the point of breaking
- Excessive Lagrangian bias reduces SDM but suppresses the overturning

Key insight

- A **moderate Lagrangian bias**, combined with refinement, achieves the best balance between: **physical fidelity** and **low spurious mixing**

Conclusions and outlook I

- The variational ALE movement gives a compact way to combine three goals in one solve: Lagrangian dynamics, mesh smoothing, and gradient-tracking refinement.
- In the internal-wave benchmarks, SLS captures propagation, bottom effects, and breaking while keeping SDM under control.
- Analysis showed that the **monitor function** approach is especially effective in reducing SDM without suppressing vertical mass transfer.

Future work includes larger-scale ocean configurations, Coriolis effects, turbulence / mixing closures, and additional optimality criteria tailored to realistic ocean applications.

Acknowledgments I

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