Loading non-Maxwellian velocity distributions in particle-in-cell (PIC) simulation

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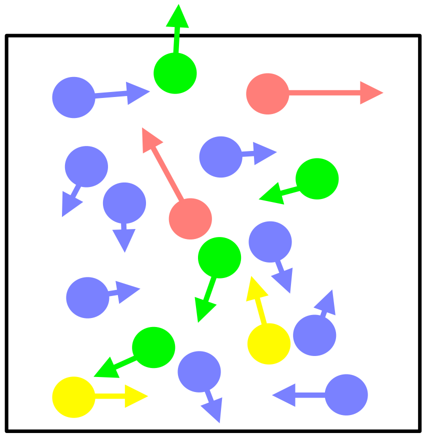
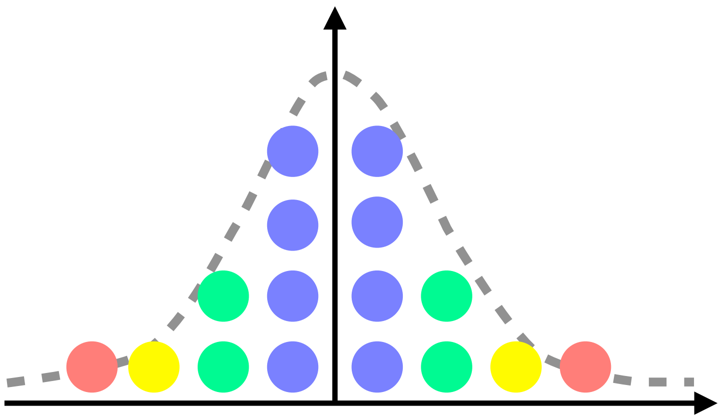
Velocity distribution in PIC simulation





- Maxwell-Boltznmann distribution

$$f(\boldsymbol{v})d^3v = N\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} \exp\left(-\frac{mv^2}{2T}\right)d^3v$$

- Box=Muller method (Box & Muller 1958)
  - Two uniform random variates: $U_1, U_2 \subset (0,1)$

$$n_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2)$$
$$n_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$$

How to start from non-Maxwellian velocity distributions?

Monte-Carlo generation of VDFs

- General methods - often inefficient
  - Inverse transform method
  - Acceptance-rejection method

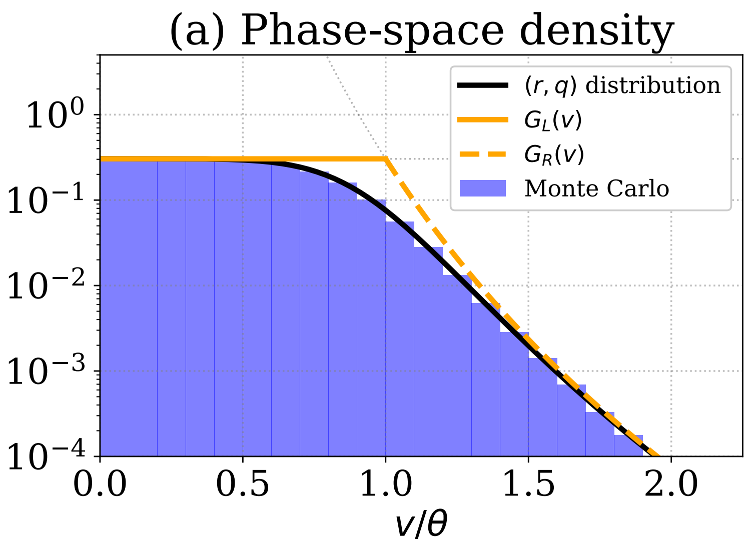
- Kappa & relativistic distributions (SZ & Nakano 2022 PoP)
- Loss-cone distributions (SZ & Nakano 2023 JGR)
- Nine distributions (SZ, Usami, Matsukiyo 2026 JGR)
  - 1. (r,q) distribution
  - 2. Regularized Kappa distribution
  - 3. Subtracted Kappa distribution
  - 4 & 5. Ring and shell distributions
  - 6 & 7. Ring and shell Maxwellians
  - 8. Super Gaussian distribution
  - 9. Filled-shell distribution

1. (r,q) distribution

[Zaheer+ 2004, Qureshi+ 2004]

$$f_{\rm rq}(\boldsymbol{v})d^3v = N \cdot C_{\rm rq}\left(1 + \frac{1}{q-1}\left(\frac{v_\parallel^2}{\theta_\parallel^2} + \frac{v_\perp^2}{\theta_\perp^2}\right)^{1+r}\right)^{-q} d^3v,$$

$$C_{\rm rq} = \frac{3\Gamma(q)}{4\pi(q-1)^{\frac{3}{2(1+r)}}\theta_\parallel\theta_\perp^2\Gamma\left(1+\frac{3}{2(1+r)}\right)\Gamma\left(q-\frac{3}{2(1+r)}\right)},$$



- Generalization of
  - Kappa & flattop VDFs
- Two methods
  - (A) Piecewise rejection method
  - (B) Beta-prime method

Beta-prime method

- Generalized beta-prime distribution

$$\mathrm{B}'(x;\alpha,\beta,p,q) = \frac{\alpha p\, \Gamma(\alpha+\beta)}{q^{\alpha p}\Gamma(1+\alpha)\Gamma(\beta)} \left(1+\left(\frac{x}{q}\right)^{p}\right)^{-(\alpha+\beta)} x^{\alpha p-1}$$

- (r,q) distribution in spherical coordinates

$$f_{\rm rq}(v)4\pi v^2 = N_0 B'\left(v; \frac{3}{2(1+r)}, q-\frac{3}{2(1+r)}, \; 2(1+r), \; (q-1)^{\frac{1}{2(1+r)}}\theta\right)$$

- Beta-prime random number

$$X_{\mathrm{B}'(\alpha,\beta,p,q)} = q\left(\frac{X_{\mathrm{Ga}(\alpha,\delta)}}{X_{\mathrm{Ga}(\beta,\delta)}}\right)^{1/p}$$

Beta-prime random number

Gamma random numbers

Algorithm 3-1: Beta-prime method

generate $X_1 \sim \mathrm{Ga}(\frac{3}{2(1+r)}, 1)$

generate $X_2 \sim \mathrm{Ga}(q - \frac{3}{2(1+r)}, 1)$

generate $U_1, U_2 \sim U(0,1)$

$x \leftarrow [(q-1)X_1/X_2]^{\frac{1}{2(1+r)}}$

$v_{\parallel} \leftarrow \theta_{\parallel} x\,(2U_1 - 1)$

$v_{\perp 1} \leftarrow 2\theta_{\perp} x\sqrt{U_1(1-U_1)}\cos(2\pi U_2)$

$v_{\perp 2} \leftarrow 2\theta_{\perp} x\sqrt{U_1(1-U_1)}\sin(2\pi U_2)$

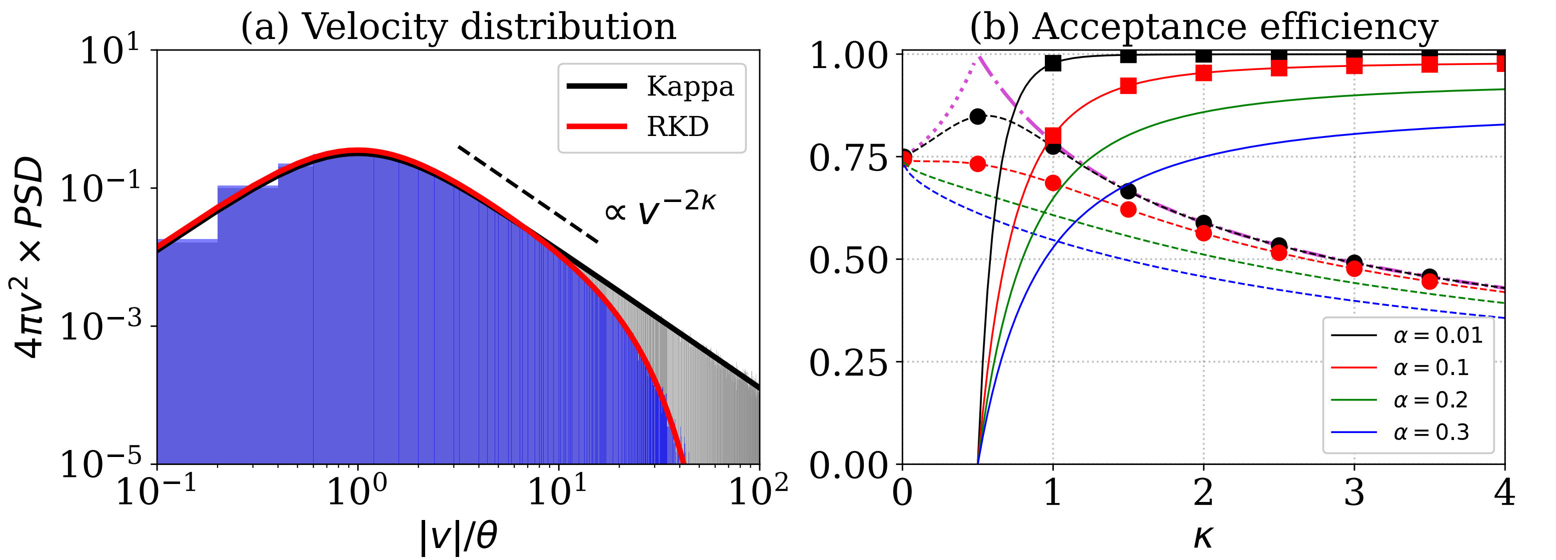
return $v_{\parallel}, v_{\perp 1}, v_{\perp 2}$

2. Regularized Kappa distribution

[Scherer+ 2017]

- Kappa distribution with a high-energy cutoff

$$f_{\mathrm{rk}}(\boldsymbol{v};\kappa,\theta,\alpha)d^3v = \frac{N_0}{(\pi\kappa\theta^2)^{3/2}\mathcal{U}\left(\frac{3}{2},\frac{3}{2}-\kappa,\alpha^2\kappa\right)}\left(1+\frac{v^2}{\kappa\theta^2}\right)^{-(\kappa+1)}\exp\left(-\alpha^2\frac{v^2}{\theta^2}\right)d^3v$$



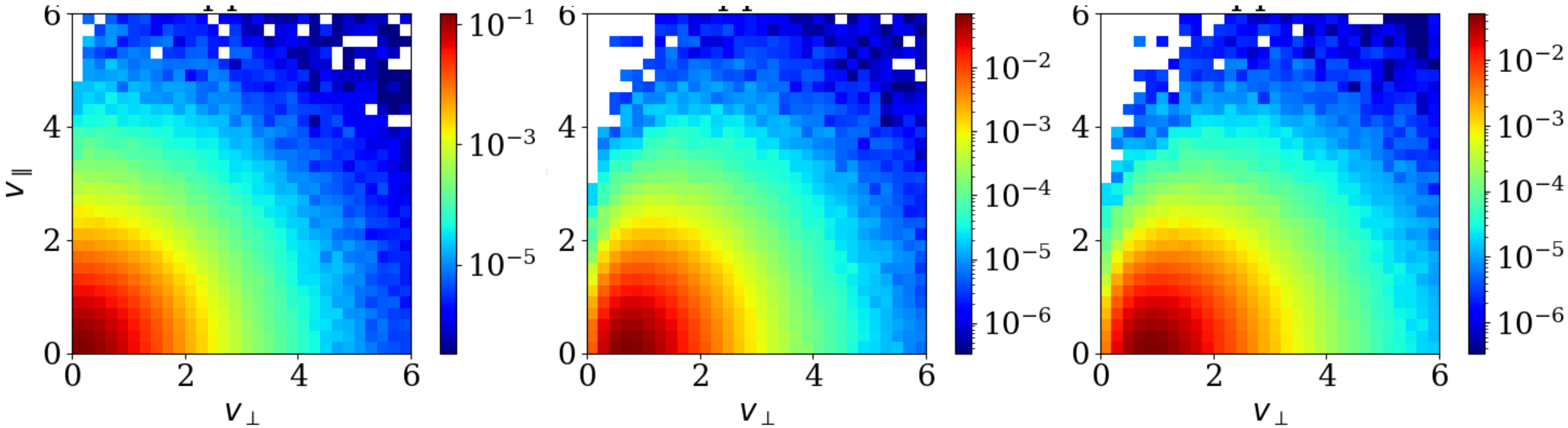
- Two rejection methods are compared

3. Subtracted Kappa distribution

[Summers & Stone 2025]

- Kappa Loss-Cone (KLC) model for a narrow loss-cone

$$f_{\rm sk}(v_\parallel, v_\perp)\, d^3v = N \cdot C_{\rm sk} \left\{ \frac{1-\Delta\beta}{1-\beta} \left(1 + \frac{{v_\parallel}^2}{\kappa\theta_\parallel^2} + \frac{v_\perp^2}{\kappa\theta_\perp^2}\right)^{-(\kappa+1)} - \frac{1-\Delta}{1-\beta} \left(1 + \frac{{v_\parallel}^2}{\kappa\theta_\parallel^2} + \frac{v_\perp^2}{\beta\kappa\theta_\perp^2}\right)^{-(\kappa+1)} \right\} d^3v$$



β=0.0 β=0.5 β=1.0

Standard Kappa distribution

Dory/Summers-type
KLC distribution with j=1

3. Subtracted Kappa distribution

[Summers & Stone 2025]

- Kappa Loss-Cone (KLC) model for a narrow loss-cone

$$f_{sk}(v_\parallel, v_\perp)\, d^3v = N \cdot C_{sk} \left\{ \frac{1-\Delta\beta}{1-\beta} \left(1 + \frac{{v_\parallel}^2}{\kappa\theta_\parallel^2} + \frac{v_\perp^2}{\kappa\theta_\perp^2}\right)^{-(\kappa+1)} - \frac{1-\Delta}{1-\beta} \left(1 + \frac{{v_\parallel}^2}{\kappa\theta_\parallel^2} + \frac{v_\perp^2}{\beta\kappa\theta_\perp^2}\right)^{-(\kappa+1)} \right\} d^3v$$

Algorithm 5: subtracted Kappa

generate $U_1, U_2, U_3 \sim U(0,1)$ — Uniform variate
generate $N \sim \mathcal{N}(0,1)$ — Normal variate
generate $Y \sim \mathrm{Ga}(\kappa - 1/2, 2)$ — Gamma variate
$x \leftarrow -\log U_1 - \beta \log\left(\min\left(\frac{U_2}{1-\Delta}, 1\right)\right)$
$v_{\perp 1} \leftarrow \theta_\perp \sqrt{2\kappa x}\cos(2\pi U_3)/\sqrt{Y}$
$v_{\perp 2} \leftarrow \theta_\perp \sqrt{2\kappa x}\sin(2\pi U_3)/\sqrt{Y}$
$v_\parallel \leftarrow \theta_\parallel \sqrt{\kappa} N/\sqrt{Y}$
return $v_{\perp 1}, v_{\perp 2}, v_\parallel$

- Loss-cone filling factor Δ is included
- Formal proof in our JGR paper
- See also Min+ 2025

4~7. Ring and shell distributions

[Usami & Horiuchi 2022 FrP, SZ+ 2026 JGR]

- ## Conventional ring/shell distributions
  - Gaussian width, but recipes are not so simple

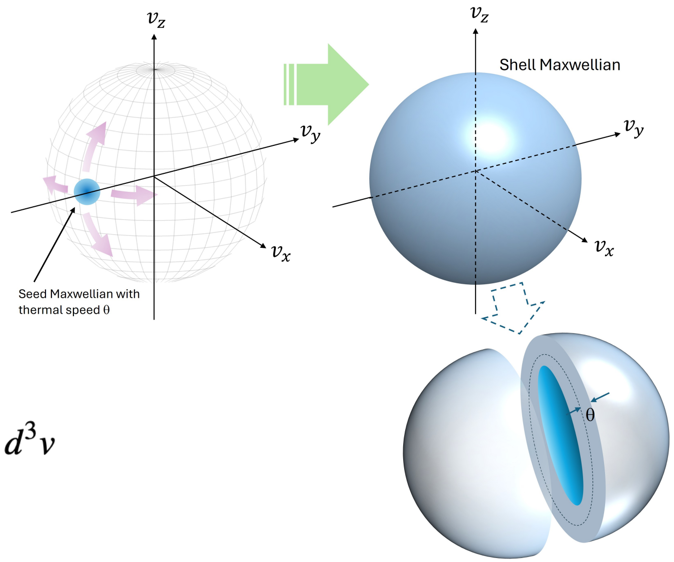
$$f_{\rm ring}(\boldsymbol{v})d^3v = N_0 \frac{1}{\pi^{3/2}\theta_{\parallel}\theta_{\perp}^2 C_2\left(\frac{V}{\theta_\perp}\right)} \exp\left(-\frac{v_{\parallel}^2}{\theta_{\parallel}^2} - \frac{(v_\perp - V)^2}{\theta_\perp^2}\right) d^3v$$

$$C_2(x) = \exp(-x^2) + \sqrt{\pi}x\,\mathrm{erfc}(-x)$$

$$f_{\rm shell}(\boldsymbol{v})d^3v = \frac{N_0}{2\pi\theta^3 C_3\left(\frac{V}{\theta}\right)} \exp\left(-\frac{(|v| - V)^2}{\theta^2}\right) d^3v$$

$$C_3(x) = x\exp(-x^2) + \sqrt{\pi}\left(x^2 + \frac{1}{2}\right)\mathrm{erfc}(-x).$$

- ## Ring/shell Maxwellians
  - Based on scattering of a seed population

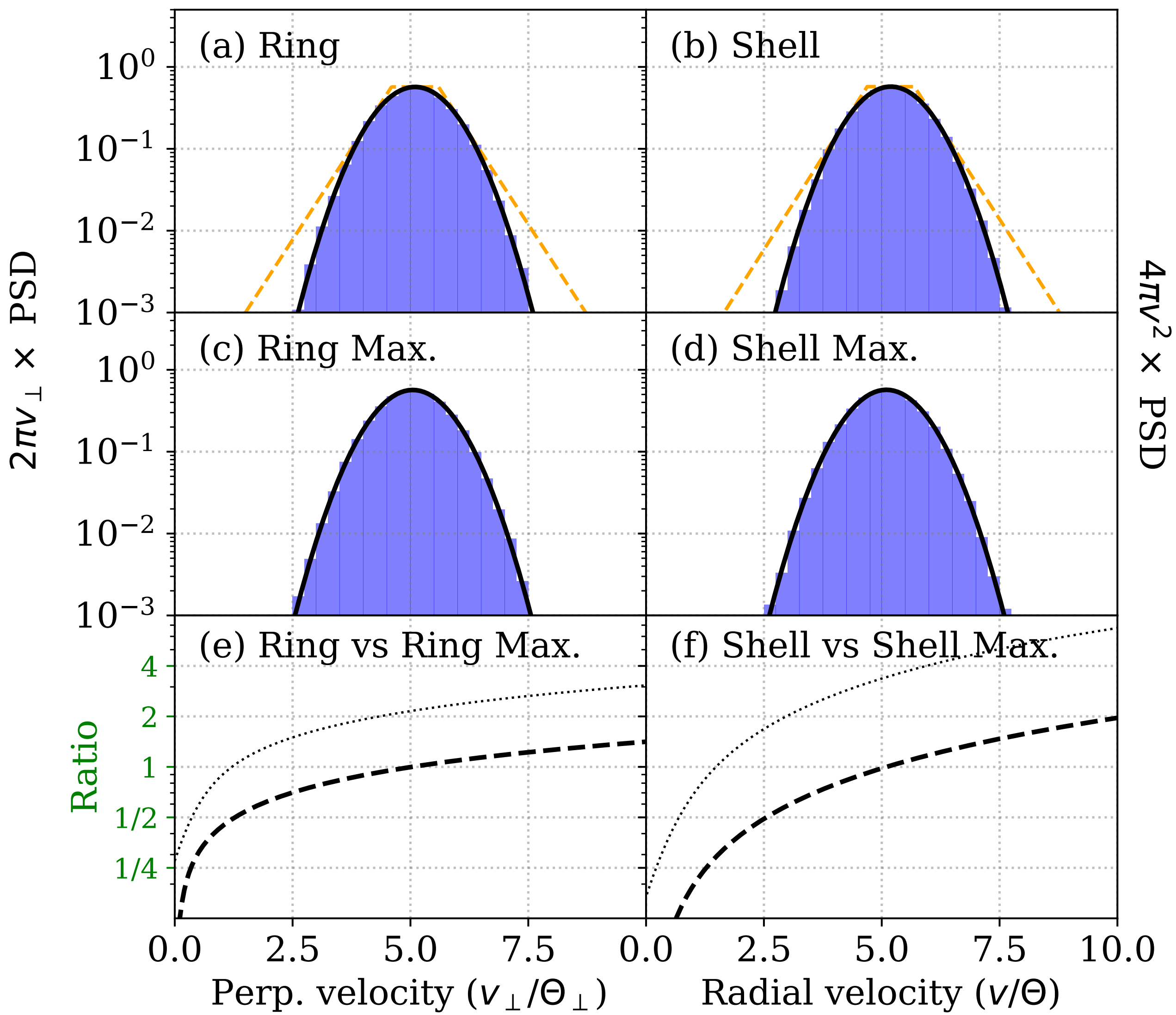


$$f_{\rm rM}(\boldsymbol{v})d^3v = \frac{N_0}{\pi^{3/2}\theta_{\parallel}\theta_{\perp}^2} \exp\left(-\frac{v_{\parallel}^2}{\theta_{\parallel}^2} - \frac{v_\perp^2 + V^2}{\theta_\perp^2}\right) I_0\left(\frac{2v_\perp V}{\theta_\perp^2}\right) d^3v$$

$$f_{\rm sM}(\boldsymbol{v})d^3v = \frac{N_0}{4rV\theta(\pi)^{3/2}} \left\{\exp\left(-\frac{(r-V)^2}{\theta^2}\right) - \exp\left(-\frac{(r+V)^2}{\theta^2}\right)\right\} d^3v$$

[NEW]

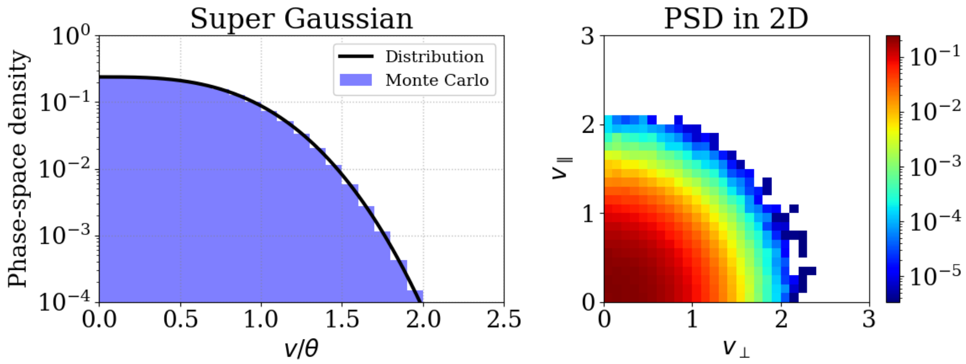
4~7. Ring & shell distributions



8 & 9. Isotropic distributions

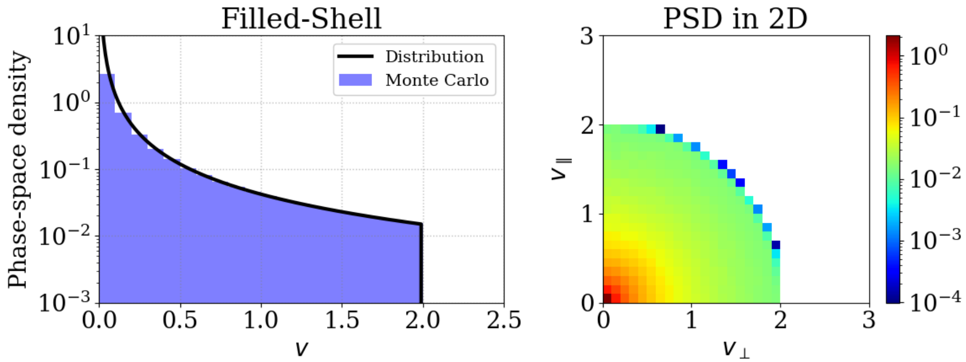
- Super Gaussian (Dum 1974)

$$\frac{Np}{4\pi\theta^3\Gamma(3/p)} \exp\left\{-\left(\frac{v}{\theta}\right)^p\right\}$$



- Filled-Shell distribution (popular in heliophysics)

$$\frac{N(3+p)}{4\pi V^{3+p}} v^p \cdot \mathcal{H}(V-v)$$



- For recipes, please see our JGR paper

Summary

- Numerical recipes for nine VDFs
  - 1. (r,q) distribution (including flattop)
  - 2. Regularized Kappa distribution
  - 3. Subtracted Kappa distribution
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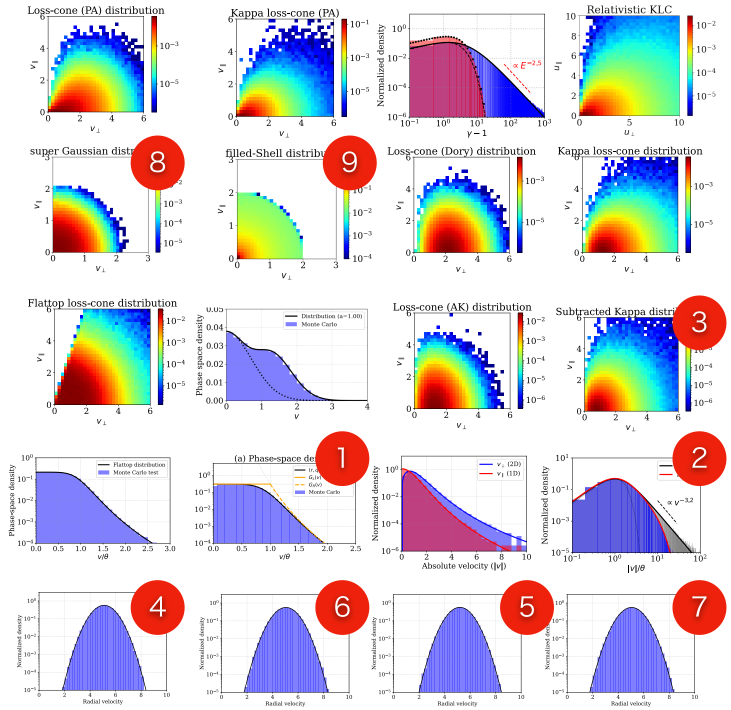

- Reference:
  - S. Zenitani, S. Usami, & S. Matsukiyo, JGR: Space Physics, 131, e2025JA034669 (2026)
  - https://doi.org/10.1029/2025JA034669

- Monte Carlo chapter of a new PIC textbook?