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Accurate numerical solution and analytical approximation for the wind profile over flat terrain

Ulf Janicke, Lutz Janicke
Janicke Consulting – Environmental Physics
Überlingen, Germany

Motivation

Pre-defined vertical profiles of wind speed and wind direction are required, among other, for Gaussian and Lagrangian dispersion models.

- A numerical procedure must be fast and accurate (1D prognostic wind field model for the poor).
- An analytical formulation must be valid at larger altitudes and should describe profiles of speed and direction consistently.
- 1D profiles should allow a smooth transition to 3D prognostic models.

Starting point: Equations of Motion

The equations of motion read

$$\frac{\partial}{\partial z}\left(K_m \frac{\partial u}{\partial z}\right) = -f_c(v - v_g) \tag{1}$$

$$\frac{\partial}{\partial z}\left(K_m \frac{\partial v}{\partial z}\right) = f_c(u - u_g) \tag{2}$$

with exchange coefficient K_m, Coriolis parameter f_c, geostrophic wind components u_g and v_g, and horizontal wind components u and v.

The boundary conditions are

$$u(0) = 0 \tag{3}$$

$$v(0) = 0 \tag{4}$$

$$u(z \to \infty) = u_g \tag{5}$$

$$v(z \to \infty) = v_g \tag{6}$$

STEP 0: Use scaled, complex variables

$$\mu = \tilde{u} + i\tilde{v} \tag{7}$$

The equations of motion then read

$$(k\mu')' = i(\mu - 1) \tag{8}$$

with the boundary conditions

$$\mu(0) = 0 \tag{9}$$

$$\mu(\tilde{z} \to \infty) = 1 \tag{10}$$

For $k = \text{const} = k_0$ this reduces to the oscillator equation

$$\omega^{-2}\mu'' - \mu = -1 \tag{11}$$

with $\omega^2 = i/k_0$. Solution is the Ekman spiral

$$\mu(\tilde{z}) = 1 + b\exp(-\omega\tilde{z}) \tag{12}$$

Putting work into context

$$(k\mu')' = i(\mu - 1)$$

Numerical and approximate analytical solutions for this equation are well known (see standard text books).

Maybe some of the tricks and steps presented in the following are not so well known and turn out to be useful.

Numerical Integration

STEP 1: At larger heights $\tilde{z} > \hat{z}$, where k is very small, assume $k = \text{const}$. Here the Ekman solution applies with some constant b and integration up to $\tilde{z} \to \infty$ is avoided.

For $\tilde{z} \leq \hat{z}$, Eq. (8)

$$(k\mu')' = i(\mu - 1) \quad (13)$$

is solved by

$$\mu(\tilde{z}) = 1 + a_1\mu_1(\tilde{z}) + a_2\mu_2(\tilde{z}) \quad (14)$$

with the homogeneous solutions $\mu_{1,2}$.

The constants b, a_1, a_2 are fixed by the condition $\mu(0) = 0$ and continuity of μ and μ' at $\tilde{z} = \hat{z}$.

STEP 2: Use for good numerical accuracy the boundary conditions

$$\mu_1(0) = 0 \quad (15)$$

$$\mu_1'(0) = 1 \quad (16)$$

(integration from bottom to top) and

$$\mu_2(\hat{z}) = 0 \quad (17)$$

$$\mu_2'(\hat{z}) = 1 \quad (18)$$

(integration from top to bottom).

STEP 3: carry out integration for $\mu_{1,2}$ numerically e.g. by Runge-Kutta.

Analytical Approximation

STEP 1: Assume $k = \text{const} = k_0$ above some height $\tilde{z}_1$.

The result is the analytical Ekman spiral for $\tilde{z} > \tilde{z}_1$ with some constant b and some value for the geostrophic wind γ.

STEP 2: Use one of the commonly applied analytical wind speed profiles for $\tilde{z} \leq \tilde{z}_1$ with a constant turn of wind direction a.

Fix constants b and γ by the condition of continuity of μ and μ' at $\tilde{z} = \tilde{z}_1$.

STEP 3: Empirical setting of a, $\tilde{z}_1$, k_0 by comparisons with the accurate numerical solution.

Analytical Approximation – Explicit Form

Lower layer $z \le h_1$:

$$u(z) = u_1(z)\cos\left[\alpha_\mathrm{a} + a(z - h_\mathrm{a})\right] \quad (19)$$

$$v(z) = u_1(z)\sin\left[\alpha_\mathrm{a} + a(z - h_\mathrm{a})\right] \quad (20)$$

Upper layer $z > h_1$:

$$u(z) = u_1(h_1)c_1 + \frac{1}{2A}\left[(1 - c(z))\,p + s(z)q\right] \quad (21)$$

$$v(z) = u_1(h_1)s_1 + \frac{1}{2A}\left[(c(z) - 1)\,q + s(z)p\right] \quad (22)$$

with

$$c_1 = \cos\left[\alpha_\mathrm{a} + a(h_1 - h_\mathrm{a})\right] \quad (23)$$

$$s_1 = \sin\left[\alpha_\mathrm{a} + a(h_1 - h_\mathrm{a})\right] \quad (24)$$

$$p = u_1'(h_1)w_+ + a u_1(h_1)w_- \quad (25)$$

$$q = u_1'(h_1)w_- - a u_1(h_1)w_+ \quad (26)$$

$$w_+ = c_1 + s_1 \quad (27)$$

$$w_- = c_1 - s_1 \quad (28)$$

$$c(z) = \exp\left[-A(z - h_1)\right]\cos\left[A(z - h_1)\right] \quad (29)$$

$$s(z) = \exp\left[-A(z - h_1)\right]\sin\left[A(z - h_1)\right] \quad (30)$$

$$A = \sqrt{|f_\mathrm{c}|/2K_0} \quad (31)$$

Eqs. (21) and (22) apply to $f_\mathrm{c} > 0$ (northern hemisphere). For $f_\mathrm{c} < 0$ (southern hemisphere)

$$u(z) = u_1(h_1)c_1 + \frac{1}{2A}\left[(1 - c(z))\,q + s(z)p\right] \quad (32)$$

$$v(z) = u_1(h_1)s_1 - \frac{1}{2A}\left[(c(z) - 1)\,p + s(z)q\right] \quad (33)$$

$$u_1(z) = \frac{u_*}{\kappa}\begin{cases} \Psi_0\left(\frac{z}{L}\right) & \text{for } 1/L \ge 0 \\ \left[\ln\left(\frac{z+z_0}{z_0}\right) - \Psi_1\left(\frac{z}{L}\right)\right] & \text{for } 1/L < 0 \end{cases} \quad (34)$$

$$\Psi_0 = \ln\left(\frac{z + z_0}{z_0}\right) + 5\left(\frac{z}{L}\right) \quad (35)$$

$$\Psi_1 = \ln\left[\left(\frac{1 + X}{1 + X_0}\right)^2 \frac{1 + X^2}{1 + X_0^2}\right] - 2\left(\mathrm{atan}X - \mathrm{atan}X_0\right) \quad (36)$$

$$X = \left(1 - 15\frac{z + z_0}{L}\right)^{1/4}, \quad X_0 = \left(1 - 15\frac{z_0}{L}\right)^{1/4} \quad (37)$$

$$K_\mathrm{m}(z) = \kappa u_*(z + z_0) \times \begin{cases} \frac{1}{1+5(z+z_0)/L}\, e^{-6\alpha z/h_\mathrm{m}} & \text{for } 1/L \ge 0 \\ \left[e^{-24\alpha z/h_\mathrm{m}} + 15\left(\frac{-(z+z_0)}{L}\right)\left(1 - 0.8\frac{z}{h_\mathrm{m}}\right)^8\right]^{1/4} & \text{for } 1/L < 0 \end{cases} \quad (38)$$

$$h_1 \approx \begin{cases} \frac{L}{20}\left[\left(1 + \frac{10h_\mathrm{m}}{3\alpha L}\right)^{1/2} - 1\right] & \text{for } 1/L \ge 0 \\ \frac{h_\mathrm{m}}{12\alpha} & \text{for } 1/L < 0 \end{cases} \quad (39)$$

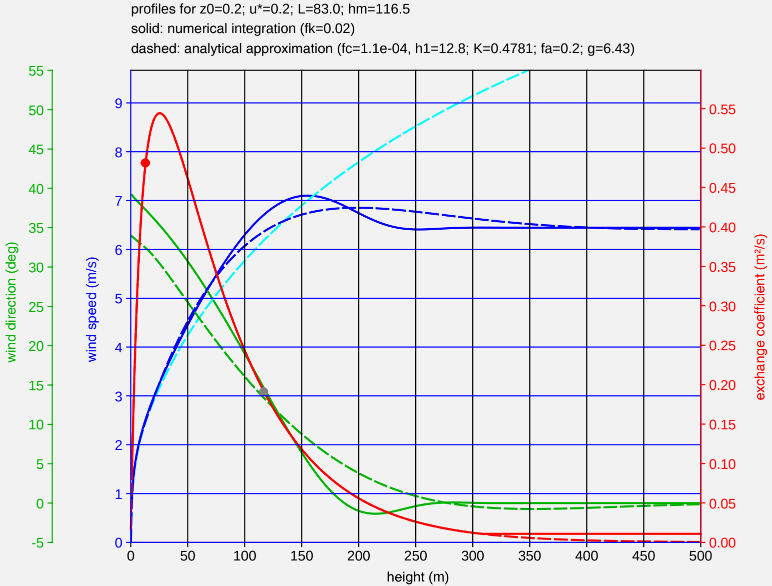
$$K_0 = K_\mathrm{m}(h_1) \quad (40)$$

$$a = \pm 0.2A \quad \text{(southern/northern hemisphere)} \quad (41)$$

Example curves

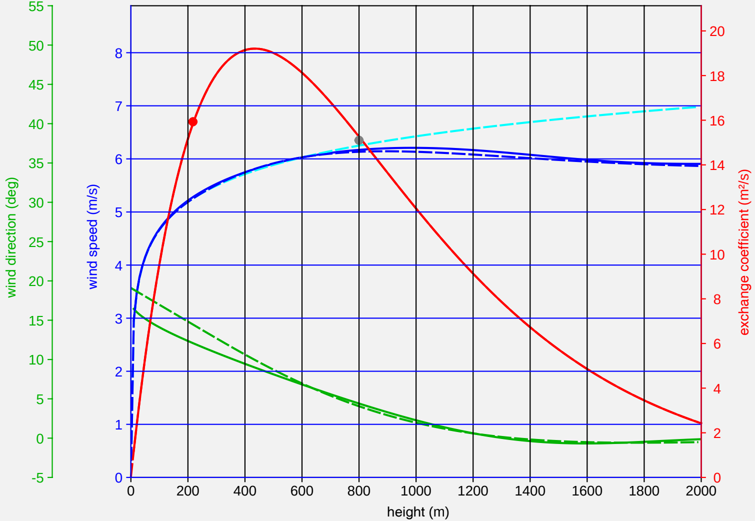
- Red curve: Exchange coefficient (red dot: h_1, gray dot: mixing height).
- Blue curves: Wind speed (solid: numerical, dashed: analytical).
- Green curves: Wind direction (solid: numerical, dashed: analytical).
- Cyan curve: Commonly used analytical profile (log-like).

Example Stable Stratification



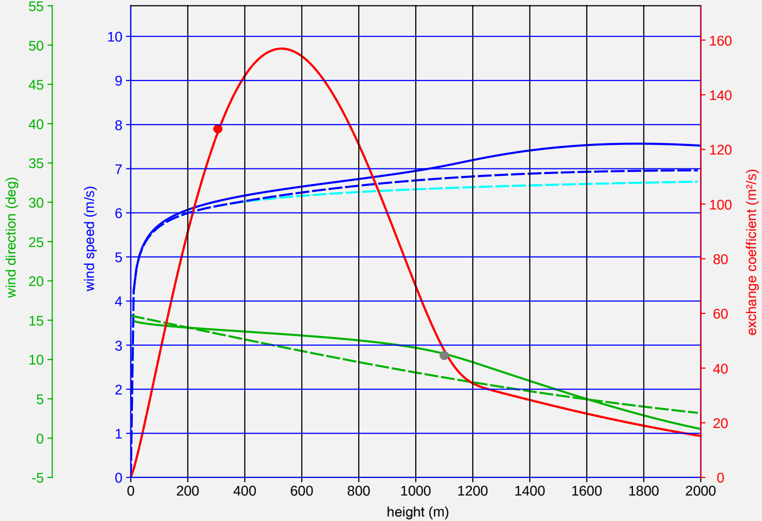
Example Neutral Stratification

profiles for z0=0.2; u*=0.3; L=99999.0; hm=800.0
solid: numerical integration (fk=0.02)
dashed: analytical approximation (fc=1.1e-04, h1=217.5; K=15.8415; fa=0.2; g=5.85)



Example Unstable Stratification

profiles for z0=0.2; u*=0.5; L=-34.0; hm=1100.0
solid: numerical integration (fk=0.02)
dashed: analytical approximation (fc=1.1e-04, h1=305.6; K=126.3056; fa=0.2; g=6.63)



Application

- Profiles were compared with a commonly used 3D prognostic model (METRAS) and with measurements.
- Profiles are used in the boundary layer model defined in German guideline VDI 3783 Part 8 (German/English, www.vdi.de). Example implementation (JAVA) will be provided.
- Profiles are implemented in the Lagrangian dispersion model LASAT (www.janicke.de) and will be applied in the regulatory model AUSTAL2000 (www.austal2000.de).

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Genaue numerische Lösung und analytische Näherung für das Windprofil über ebenem Gelände

Ulf Janicke, Lutz Janicke

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Ingenieurbüro Janicke, Hermann-Hoch-Weg 1, 88662 Überlingen, Germany