The Lidov-Kozai oscillation and Hugo von Zeipel

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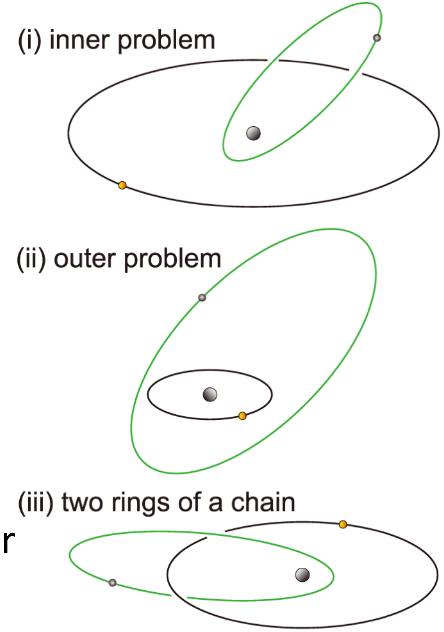
It is widely accepted that the theoretical framework of the so-called Lidov-Kozai oscillation was established independently in the early 1960s by a Soviet Union dynamicist (Michail L'vovich Lidov) and by a Japanese celestial mechanist (Yoshihide Kozai). A large variety of studies has stemmed from the original works by Lidov and Kozai, now having the prefix of "Lidov-Kozai" or "Kozai-Lidov". However, from a survey of past literature published in late 19th to early 20th century, we have confirmed that there already existed a pioneering work using a similar analysis of this subject established in that period. This was accomplished by a Swedish astronomer, Edvard Hugo von Zeipel. In this presentation we make a brief summary of von Zeipel's work on this subject in contrast to the works of Lidov and Kozai, and show that von Zeipel's achievements in the early 20th century (written and published in French under the title "*Sur l'application des séries de M. Lindstedt à l'étudedu mouvement des comètes périodiques*") already comprehended most of the fundamental and necessary formulations that the Lidov-Kozai oscillation requires. By comparing the works of Lidov, Kozai, and von Zeipel along this line of studies, we assert that the prefix "von Zeipel-Lidov-Kozai" should be used for designating this theoretical framework, and not just Lidov-Kozai or Kozai-Lidov.

For more detail, read our meep.2019.00701.0001 and von Zeipel (1910)

Definitions used in this presentation

a: semimajor axis
e: eccentricity
i: inclination

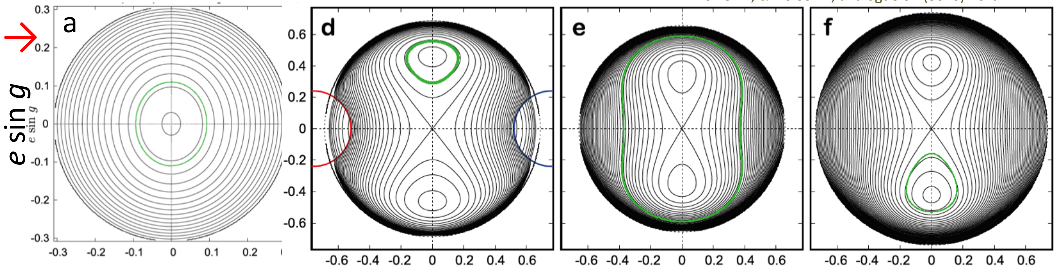
- The (classical) Lidov-Kozai oscillation in CR3BP
  - It is a series of phenomenon appearing in the circular restricted three-body problem (CR3BP)
  - The doubly-averaged CR3BP is integrable → $k^2 = (1 - e^2) \cos^2 i$ (= Lidov's c_1) is conserved
  - When the initial value of k^2 is smaller than a certain value, argument of pericenter g of the perturbed body stops circulation, and begins libration around stationary points
  - The inclination i when k^2 takes the "certain value" → Critical inclination (a function of a)
- The inner and outer problems
  - (i) The perturbed body orbits inside the perturber → the inner problem
  - (ii) The perturbed body orbits outside the perturber → the outer problem
  - (iii) The two orbits intersect, and acquire the geometry of rings of a chain
- Quadrupole level approximation
  - $\alpha = a/a' \ll 1$ (for the inner problem), $\alpha' = a'/a \ll 1$ (for the outer problem)
    - a and a' → semimajor axis of the perturbed and perturbing bodies, respectively
  - Boundary between g's libration and circulation becomes clearer using another parameter devised by Lidov, $c_2 = e^2 (2/5 - \sin i^2 \sin g^2)$: $c_2 < 0 \Leftrightarrow$ libration

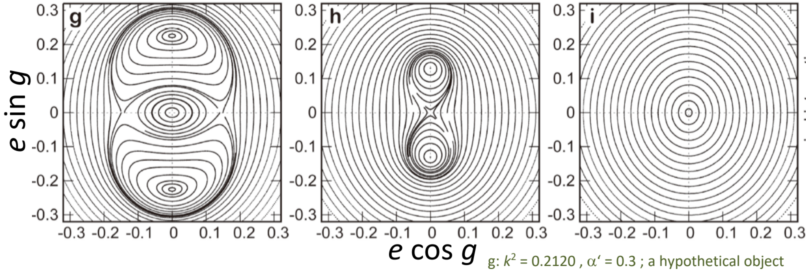


Equi-potential trajectories on polar coordinates

- The doubly-averaged CR3BP is integrable (i.e., the degrees of freedom = 1)
  - Behavior of the perturbed body is governed by a function $R(e,g)$ ← averaged disturbing function
- The polar coordinates used in this presentation → ($e \cos g$, $e \sin g$)
  - Example trajectories in the inner problem →
    - Black lines → trajectories by numerical quadrature
    - Green lines → trajectory by numerical integration
    - Stationary libration points exist at $g = \pm\pi/2$ (a & e: circulation. d & f: libration)
  - Example trajectories in the outer problem →
    - Black lines → the trajectories by numerical integration
    - Stationary libration points exist at $g = \pm\pi/2$ (in g and h)
    - Two topological boundaries exist (between g – h – i)
- Achievement of [Lidov (1961)] and [Kozai (1962)]
  - Both worked on the inner problem. Neither worked on the outer problem
  - Kozai went into higher-order approximation. He also employed equi-potential trajectories
  - Lidov detailed system's behavior on the border between circulation and libration ("Lidov diagram")

a: k^2 = 0.9045 , α = 0.3726 ; analogue of (4690) Strasbourg
d: k^2 = 0.5325 , α = 0.6569 ; analogue of (1373) Cincinnati
e: k^2 = 0.5979 , α = 0.5123 ; analogue of (1036) Ganymed
f : k^2 = 0.452 , α = 0.354 ; analogue of (3040) Kozai

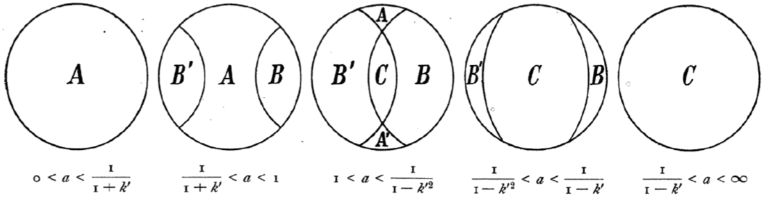




g: k^2 = 0.2120 , α' = 0.3 ; a hypothetical object
h: k^2 = 0.2185 , α' = 0.3 ; a hypothetical object
i: k^2 = 0.2400 , α' = 0.3 ; a hypothetical object

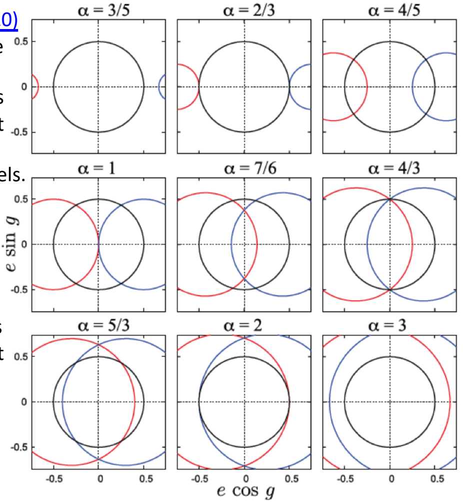
Outline of the work by von Zeipel (1910)

- Introducing the Lindstedt series into canonical perturbation theory
  - In particularly, applying the series to the doubly-average CR3BP to locate possible local extremums of the (averaged) disturbing function in addition to the origin ($e \cos g$, $e \sin g$) = (0, 0) (von Zeipel 1916a, 1916b, 1917a, 1917b)
  - It is probably a "work product" for his later famous works, currently known as the von Zeipel method
- Dealing with both the inner and outer problems
  - His subject was the motion of (long-period) comets
- Consideration of the case of orbit intersection
  - Some general discussions are given
- Approximation higher than quadrupole level
  - Making his solutions as accurate as modern level
- Use of equi-potential trajectories (see slide 6)
  - Although handwritten, they look quite accurate
- Prediction of stable orbits of small bodies
  - "unintentional" prediction of TNOs (Centaurs) ?



↗ Transcription of Figures 1-5 of von Zeipel (1910) showing how the orbit intersection curves divide phase space on the ($e \cos g$, $e \sin g$) plane. von Zeipel illustrated five cases along with the values of $\alpha = a/a'$. Note that von Zeipel's a is equivalent to α. The parameter k' indicates the maximum value of e, and $k' = 1/2$ is common to all the panels.
(Credit: John Wiley and Sons. Reproduced with permission)

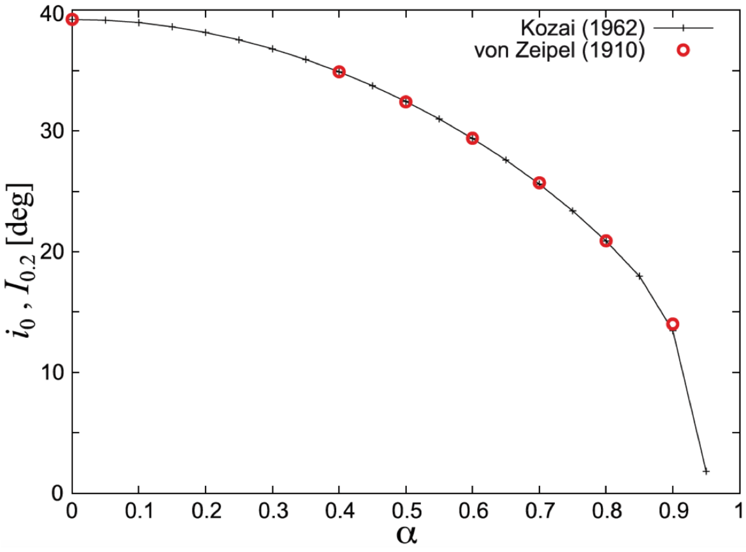
We depicted the geometry described above → with several more values of α. Note that when $\alpha > 1$, $\alpha' = \alpha^{-1} < 1$. The red and the blue partial circles represent the conditions where the orbits of the perturbed and perturbing bodies intersect each other at the ascending node (red) and at the descending node (blue) of the perturbed body. The parameter $k' = 1/2$ is common to all the panels, consistent to von Zeipel's figures.



Comparison: von Zeipel's work and modern works

(1) Critical inclination $I_{0.2}$ for the inner problem

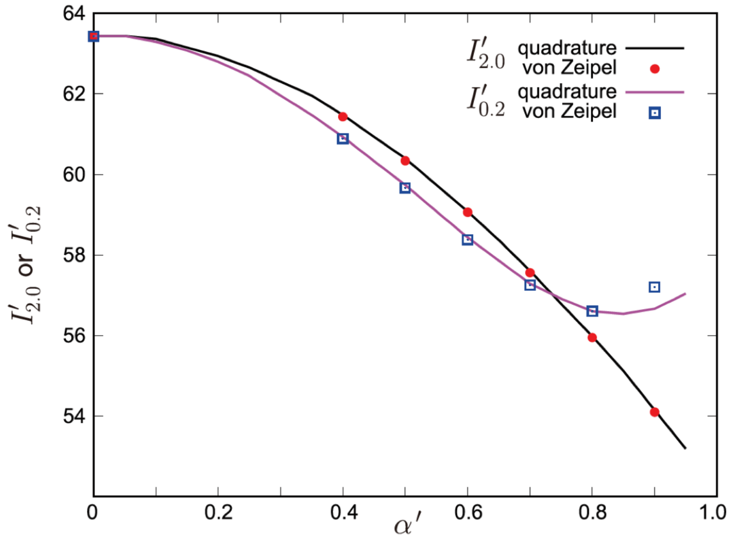
- $I_{0.2} = \cos^{-1}(3/5)^{1/2}$ when $\alpha << 1$ (quadrupole).
- $I_{0.2}$ gets monotonically smaller as α increases.
- von Zeipel's (1910) calculation results of $I_{0.2}$ (○) agrees very well with Kozai's (1962) i_0 obtained from the numerical quadrature from $\alpha = 0$ up to $\alpha = 0.9$ (—— +).



(conventional classics)

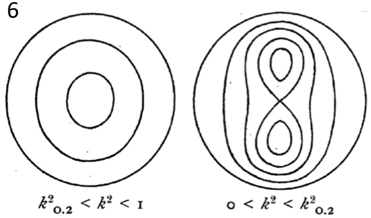
(2) Critical inclinations $I'_{0.2}$ and $I'_{2.0}$ for the outer problem

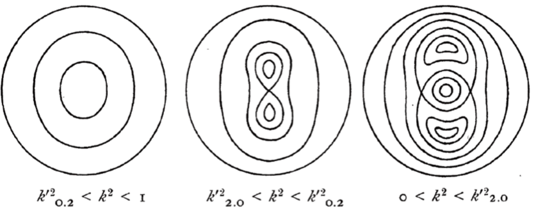
- The outer problem is more complicated than the inner one.
- Two kinds of critical inclination exist ($I'_{0.2}$ and $I'_{2.0}$).
- Both $I'_{0.2}$ and $I'_{2.0} = \cos^{-1}(1/5)^{1/2}$ when $\alpha' << 1$ (quadrupole).
- $I'_{0.2}$ has a local minimum at $\alpha' \sim 0.85$ ($I'_{0.2} > I'_{2.0}$ when $\alpha' > 0.74$).
- von Zeipel's calculation results (●, □) agrees well with our numerical quadrature up to $\alpha' \sim 0.9$ (—— and ——).



- Lidov (Лидов), М. Л. (1961), Искусственные спутники Земли, **8**, 5-45.
- Kozai, Y. (1962), The Astronomical Journal, **67**, 591–598.

Summary





← Schematic equi-potential trajectries for the inner problem on (e cos g, e sin g) that von Zeipel depicted. $k^2_{0.2}$ is the topological boundary value of k^2 (a function of α; $k^2_{0.2}$ = 3/5 when α << 1).
↙ Schematic equi-potential trajectries for the outer problem. $k'^2_{0.2}$ and $k'^2_{2.0}$ are the topological boundary values of k^2 (they are functions of α', and both are 1/5 when α' << 1).
(Credit: John Wiley and Sons. Reproduced with permission)

Conclusion - The prefix
von Zeipel-Lidov-Kozai
should be used,
not Lidov-Kozai or just Kozai

- The theoretical framework of the so-called Lidov–Kozai oscillation was already established in the early 20th century almost at the same depth or deeper by von Zeipel (1910). He dealt with not only the inner CR3BP but also the outer CR3BP in very detail.
- The fundamental formulas were completed and the major result was obtained even earlier at the end of 19th century (von Zeipel 1898; von Zeipel 1901).

A comparison of the achievements shown in Kozai's, Lidov's, and von Zeipel's work. As for the numbered notes (1-10), take a look at Table 5 of our article (p. 85).

	Kozai	Lidov	von Zeipel
General features and treatment of CR3BP			
Hamiltonian formalism	○	—	○
Gauss's form of equations	—	○	—
single averaging	—	○	—
double averaging	○	○	○
conservation of $(1 - e^2)\cos^2 i$	○	○	○
conservation of total energy	○	○	○
numerical quadrature	○	—	○
direct numerical integration	—	○	—
Doubly averaged inner CR3BP			
libration of g around $\pm\frac{\pi}{2}$	○	○	○
conservation of c_2-like variable	—	○	—
solutions in special cases	△[1]	○	—
time-dependent analytic solution	○	△[2]	—
equi-potential contours	○	△[3]	○[6]
treatment when α is not small	○	—	○
mention of actual objects	○	△[4]	○[7]
oblateness of central mass	—	△[5]	—
Doubly averaged outer CR3BP			
libration of g around $\pm\frac{\pi}{2}$	—	—	○
libration of g around 0 or π	—	—	○
equi-potential contours	—	—	○[6]
treatment when α' is not small	—	—	○
mention of actual objects	—	—	○[8]
Other features and treatment			
Treatment of orbit intersection	—	—	○[9]
Multiple perturbing bodies	—	—	△[10]

(von Zeipel's original articles)

- von Zeipel, H. (1898), Sur la forme génerale des éléments elliptiques dans le problème des trois corps, *Bihang till Kongl. Svenska Vetenskaps-Akademiens Handlingar*, 24, Afdelning I (8), 1–51.
- von Zeipel, H. (1901), Recherches sur l'existence des séries de M. Lindstedt, *Bihang till Kongl. Svenska Vetenskaps-Akademiens Handlingar*, 26, Afdelning I (8), 1–23.
- von Zeipel, H. (1910), Sur l'application des séries de M. Lindstedt à l'étude du mouvement des comètes périodiques, *Astronomische Nachrichten*, 183, 345–418.